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2. To develop and demonstrate the application and interpretation of systems of logic in various fields of scientific work.

3. To disseminate knowledge about systems of logic and their application.

4. To develop new systems of logic, new interpretations and new applications.

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The program of the Institute for accomplishing its objectives includes the publication of this journal. The journal is intended to link a wide variety of people in the field of computing machinery and in the field of theoretical logic. This requires a broad editorial policy and the publication of articles at a variety of levels. Scholarly papers of principle interest to professional logicians as well as articles pertaining to the development of systems for computing machinery and related material are solicited.

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Manuscripts appropriate to the purposes of the Journal and the Institute are solicited. They should be addressed to the Institute of Applied Logic, 47 West Water St., St. Paul 1, Minnesota. All manuscripts will be given prompt consideration and those accepted will be published as soon as possible. Authors will be given fifty reprints without charge and additional reprints will be supplied upon request at cost.

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INTRODUCTION

The term "Computing Systems" is used in the title of this Journal in its broadest sense. It is intended to include logical and mathematical systems as well as structures for machines designed to solve problems that involve computing. Thus the principal purpose of the Journal is to provide a common meeting ground, a channel for communication, in these inter-related fields. Articles will be published covering the interpretation of logical and mathematical methods in the field of computing machinery as well as theoretical technical papers in all three subjects. The editorial policy and the choice of subject matter will be substantially guided by the interest shown and communications from readers will be welcomed.

THE FOUNDATIONS OF COMPUTING MACHINERY

JOHN D. GOODELL

Part I.

It is the purpose of this section to contribute toward a common understanding between logicians and digital computing machine engineers. For the benefit of those who may not be familiar with terminologies commonly used in the computing machine industry and to insure a common reference of communication, various definitions will be given. Conversely, terms and constructions well known to logicians but comparatively unfamiliar to many designers in the computing machine field will be set forth. Consequently, certain portions of the early material will be obvious to men in each field. For those who are familiar with the entire subject matter, only the suggested forms of notation will be of interest.

Definitions and Notations

The basic units of computing machine design are circuits that respond to the presence of certain input signals by producing an output signal. These elements may be said to use the criteria established by the presence or absence of various input signals to decide whether an output signal shall appear. Hence, such a unit is termed a "Decision Element," abbreviated to D.E. when convenient.

A useful term in the language of logic is "functor." A logical functor represents the linkage between arguments. In arithmetic the signs for addition, subtraction, multiplication, etc., are functors. Examples of logical functors are the terms "and," "or," "negation," etc.

The D.E. is a computing machine functor. In a binary (two valued) system one way to consider such D.E.'s is to view the inputs as variables having the value 1 or 0 and the output as a variable substituted for the bracketed inputs including the linkage of the functor.

In this article the lower case Latin letters will be used to symbolize variables. For purposes of simplifying presentations and to eliminate the need for brackets, the functor will be written ahead of the variables. Thus, using K as the sign for conjunction, Kpq is equal to pKq is equal to p and q.

There are many possible interpretations of computing structures in terms of logical systems, and for various purposes all may be useful. Several will be given but this presentation is not intended to indicate that these interpretations are limiting, or necessarily superior.

(Received Dec. 17, 1951)

They have been applied and shown to be useful. Other interpretations have been and will be developed.

The Calculus of Propositions is a basic logical system of deduction originally invented by the Stoics approximately fifty years after the time of Aristotle. The principles involved have long been used intuitively by mathematicians and were applied by John Boole in the development of the form of algebra that bears his name. In the nineteenth century the entire Calculus of Propositions was re-discovered independently by Frege, and since that time many systems of Modern Formal Logic have been constructed and elaborated on this fundamental framework.

The variables in the Calculus of Propositions are declarative propositional arguments that are considered only with regard to whether they are true or false, and these characteristics are represented respectively by 1 and 0. No internal linkages between the propositional arguments are recognized in this system.

The variables in a binary digital computing machine are also symbolized by 1 and 0 and are dynamically represented internally in the machine by the presence or absence of electrical signals. The variables in such computing machines have no internal linkages and are related only in the sense of the two values 1 and 0. It should be mentioned that the sense of the term "variable" in the logical as well as the computing machine language means that any such symbol may represent any argument or object allowed by the restrictions of the system. Thus, in the Calculus of Propositions the symbol p may represent any declarative proposition. In the computing machine structure an electrical pulse may represent an abstract number, an object, or anything else for which the system of the machine design produces valid results. For the internal operation of the machine this is of no consequence. It is important only in the exterior interpretation of the machine operations, and the results produced by the machine.

In the Calculus of Propositions there are four possible functors for one argument. These are represented by the following matrix where p is the argument, N, V, A_s and F_s are the functors Negation, Verum, Asertium and Falsum respectively, and the possible values are 1 and 0.

		N	V	As	Fs
p →	0	1	1	0	0
	1	0	1	1	0

The functor A_s is a short circuit (a connecting wire), F_s is an open circuit (a cut wire), V is a device that generates a continuous stream of output signals, and N is a similar device with facilities for canceling the output at any time when some input p is equal to 1.

For two arguments there are sixteen possible functors, each of which may be represented by a matrix where one input is p, the other is q, each has the value 1 or 0, and the intersection is the correct value for representing the two arguments bracketed with the functor. The functor is shown in the upper left-hand corner. The following is such a matrix for the functor K, which is used to symbolize conjunction (and).[1]

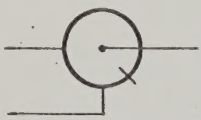
q

	K	0	1
p	0	0	0
	1	0	1

It will be seen that the four possible output conditions may be related to the four quadrants of a circle. A notation based on this is proposed for block diagram drafting, programming and general discussion of D.E.'s.[2] In this notation two inputs and one output are provided, and the characteristics of the functor are indicated by strokes through the circumference of the circle in the appropriate quadrants.

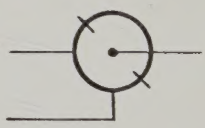
K	0	1
0	0	0
1	0	1

=



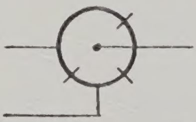
E	0	1
0	1	0
1	0	1

=



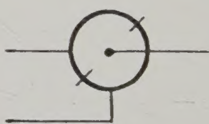
A	0	1
0	0	1
1	1	1

=



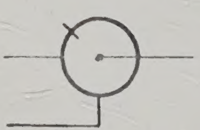
R	0	1
0	0	1
1	1	0

=



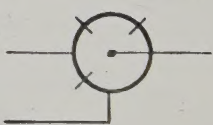
S	0	1
0	1	0
1	0	0

=

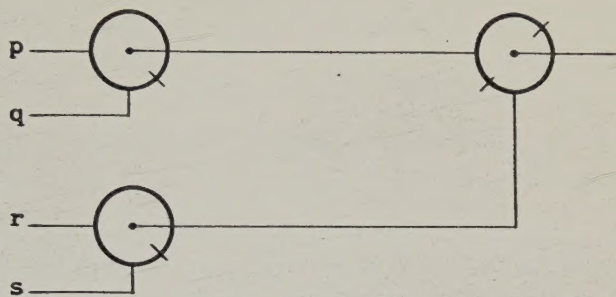


D	0	1
0	1	1
1	1	0

=



In these matrices and in the discussions that follow, E is the symbol for equivalence, A for non-exclusive or, R for exclusive or, S is the negation of A, and D is the negation of K. Using the convention of placing the functors ahead of the variables, RKpqKrs is equal to (p and q) R (r and s) and, using the suggested notation, may also be written:



Synchronization

Logical formulas are conventionally written from left to right across the page. Intelligibility requires that only one symbol be written in one space on the page. In machine design there are similar problems since signals must be distributed with respect to time. In order to accomplish this the machine may be so carefully designed with respect to the time intervals introduced by the various elements that no conflict can appear, or all shifts between elements may be made to occur in step with a central synchronizing clock. The functor V is suitable for representation of a clock since it implies the continual generation of output signals. Such a device is an oscillator or pulse generator.

A system that is not clock controlled is said to be a free running system. While this type of system is often desirable from the point of view of minimum time delays, it is difficult to design with adequate reliability.

There are two ways to maintain synchronization in a clock controlled system. One method is to construct all D.E.'s so that their action is triggered by clock pulses. In such a structure the input signals may be stored temporarily within the D.E. and the output appears when the clock trigger pulse is applied. Another method is to provide precisely timed delays within each D.E. so that the output always occurs precisely one clock pulse time after the input. In any large scale computer small errors in time delays will be cumulative, and it is ordinarily necessary to provide periodic means of resynchronization with clock controlled elements.

Definite Constructions

Certain functors may be constructed as combinations of others, and determining the optimum basic functors using any given technique is an important consideration. This involves weighing the advantages of standardization with a minimum of basic D.E.'s, the economic factors, flexibility of program facilities and other considerations. For example, from either D or S it is possible to construct any functor for one or two variables. Thus, if such a device were sufficiently simple and desirable in other respects, using a particular technique, this would mean that only one basic D.E. design would be adequate for all systems in this class. The other extreme case is to provide every type of D.E. that might be required.

Logical proofs are often useful in solving questions of this kind. Consider, for example, the problem whether it is possible to construct a complete system using only the functors K and R. One approach to this is to determine whether it is possible to construct a thesis in the Calculus of Propositions using only these two functors. If it is not possible, then there must be functors that cannot be constructed from these two. It can be proved that there is no thesis which could be determined only by K and R. In the theses required for this proof, C is the functor for implication (if - then). The following theses from the Calculus of Propositions are required for the complete proofs.

1. CKpqp
2. CRRpqrRpRqr
3. CRpRqrRqRpr
4. CRpqRqp
5. CRpRqrRrRpq
6. CRpRqRrsRpRRqrs
7. CRKpRqrsRKpqRKprs
8. CRKRqrpsRKpqRKprs

The method of proof is as follows:

a. Consider propositions using only the functor K. Then any such proposition must begin with K and be of the type Kxy.

b. There must be some thesis which is the shortest thesis of this type.

c. Using thesis 1 and substituting for p/x and for q/y, Kxy is the precedence and x is the consequence. By the rule of detachment we can now assert the thesis x.

d. This thesis x must also be of the type Kxy, but the x in this Kxy must be a shorter thesis than the thesis x that contains Kxy. Thus, successively we may deduce shorter and shorter theses x, x_1, x_2,....x_n.

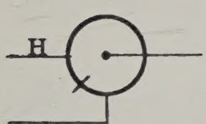
e. Finally we must deduce a shortest thesis of this type which is either Kpq or the small Latin letter p. But if we have Kpq, then using thesis 1 and the rule of detachment we may assert its consequence, which is p. Thus, the shortest thesis of this series is p.

f. But for p we can substitute either 1 or 0, and if we substitute 0 then p is false and represents a contradiction in the system of the Calculus of Propositions which contain only theses which are true.

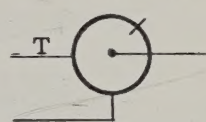
g. Hence, any thesis constructed using only K as a functor leads to a contradiction.

The proofs for R and for the combination of K and R are similar but are too lengthy to warrant inclusion here.

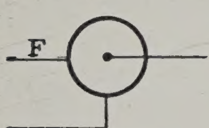
It may be finally concluded from this reasoning that using K and R it is possible to construct any of the following forms for two arguments:



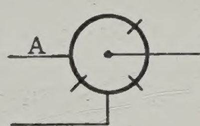
= Hpq = RpKpq



= Tpq = RqKpq



= Fpq = RKpqRpq



= Apq = RRpqKpq

and the following for one argument:

As	
0	0
1	1

Asp = p = Kpp

Fs	
0	0
1	0

Fsp = Rpp

No other functors for one or two arguments can be defined by K and R. It is not possible to define either

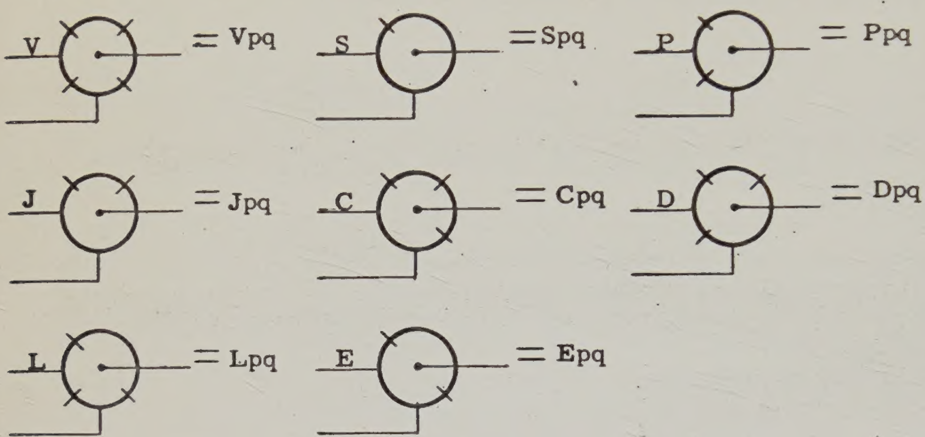
N	
0	1
1	0

= Np

Vr	
0	1
1	1

= Vrp

nor:



but any one of these functors that cannot be defined by K and R provide a complete system if added to K and R.

It has been mentioned that many computing machines will be synchronized by means of a central clock and that this device is analogous to V. Thus, in any clock synchronized computer the availability of D.E.'s having the characteristics K and R makes it possible to construct any functors for one or two arguments. The following are definitions of the nine functors that cannot be defined by K and R but can be defined with K, R and V.

(3) Np RVpp VII. Vpq RVpRpq VIII. Spq RVpRRpqKpq

IX. Ppq RVpKqVp X. Jpq RVpKpVq XI. Cpq RVpKpRVrqq

XII. Dpq RVpKpq XIII. Lpq RVpKpVq XIV. Epq RVpRpq

This means that using any technique that permits relatively simple constructions for K and R, a clock synchronized computer requires only these two D.E.'s for two valued variables.

Application of D.E.'s

Using D.E. units based on logical functors, it is possible to design and construct any digital computing machine. In the past it has been common practice to develop every digital computer as an essentially specialized design. In view of the large investment represented by

such devices, it seems inevitable that the trend toward using general purpose plug-in basic elements will be increasingly strong. In this way it is possible not only to program a given machine set up for various problems, but literally to rebuild the machine by changing the location of the D.E. units. Thus, a computing machine becomes a mounting board with a large number of sockets, a facility for varying the connections between sockets such as a plug-board designed similarly to a telephone switchboard, and a multiplicity of D.E. units. With a sufficient number of D.E.'s any digital computer may be built in this manner and periodically redesigned in accordance with new systems and new types of problems.

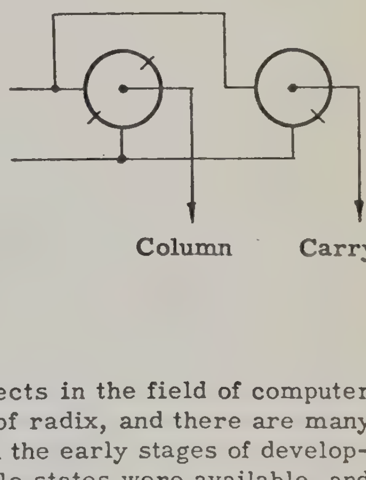
A simple example of the application of D.E. units in binary arithmetic is illustrative of the manner in which they may be used. The functor R in parallel with K establishes the required conditions. In this arrangement a signal for 1 on one line and 0 on the other produces binary sum 1 at the output of the R.D.E., and the condition where 1 appears on both lines produces 0. Where 1 appears on both lines the K.D.E. produces the output 1, which represents the necessary carry, while 0 appears at the output of K for the condition that produces 1 at the output of R.

Column (R)

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ \underline{1} & \underline{0} & \underline{0} & \underline{1} \\ 1 & 1 & 0 & 0 \end{array}$$

Carry (K)

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ \underline{1} & \underline{0} & \underline{0} & \underline{1} \\ 0 & 0 & 0 & 1 \end{array}$$



Arithmetic Base

One of the most controversial subjects in the field of computer design has been the matter of the choice of radix, and there are many different points of view. It was apparent in the early stages of development that many structures having two stable states were available, and that devices with more than two stable states were either excessively complex or unreliable. The basic question, of course, is not whether satisfactory devices for building computers in terms of radices higher than binary are available, but whether they might be superior if they were available.

No definite conclusion in this regard is presented here but a few of the arguments that have been commonly used are briefly discussed. It is pointed out that there is a strong tendency to favor binary methods on the basis of familiarity in machine design methods, and perhaps to

cling to such systems on the basis of rationalizations rather than conclusions reached after careful study. Generally, binary is viewed as the most desirable of all radices and decimal as a necessary evil in the design of equipment that must communicate with those individuals in the outside world that are not sufficiently familiar with binary notation to use it freely. Most business machine equipment is designed in decimal terms for reasons of convenience to the average user. Most large scale, high speed digital computers have been designed in binary because only binary components were available.

It is granted that there are some obscure mathematical problems that may be solved most conveniently, or perhaps only, in binary form.

In general, it seems worthwhile to consider the problem from the standpoint of coding of information, as well as in connection with the arithmetic convenience of various bases. In regard to coding, it would appear that binary is the least compact form and requires the maximum number of basic operative elements. Arithmetically, a base such as 12 with its large number of factors is attractive.

With the aid of certain reasonable hypotheses, the optimum radix in terms of current techniques has been indicated as converging on the transcendental e. This reasoning also leads to the conclusion that 2 and 4 are equally desirable and 3 may be optimum.[3]

As new techniques are developed, this entire problem will require periodic review and study.

Multiple Argument Functors

It is possible to design decision elements with multiple inputs and also multiple outputs. The latter implies a many-purpose device, the former a structure capable of effectively varying its effect throughout the program. Thus, a three input bi-value decision element may appear as one of two different functors to two of its arguments, depending on the value of the third input.

In the design of logical systems functors for three arguments ordinarily turn out to be unwieldy, but in computer design the opposite may be the case.

Variable Functors

From the logical point of view it is possible to build extremely strong systems using variable functors. From the standpoint of computing machine design, decision elements capable of dynamic control with respect to the functors they represent, constitute an important possible method of minimizing advance programming of problems.

A very high percentage of the time required for the solution of a problem in a high order computing machine is taken up by lengthy and

complicated advance steps in establishing the program. Several individuals may work for a year to establish the program for a problem that may be solved by the machine in a day of running time. Methods for establishing the programming of such problems internally on the basis of machine-made decisions are being carefully studied by a number of groups in the field, and they appear to hold forth considerable promise for future simplification and efficiency. The concept of the dynamic variable functor is potentially important in this kind of application.

Many Value Logic

In the bi-value Calculus of Propositions the symbols 1 and 0 are used to indicate that a declarative proposition is "true" or "false" without regard for any internal meaning that may be attached to individual parts of the proposition or that may link the internal structures of the proposition. This bi-value system is a very intuitive concept. It is difficult to think in terms of declarative propositions that are neither true nor false, or that may be both true and false. The problem is principally semantic. It is akin to the difficulties once encountered in thinking about zero, about negative numbers and about four dimensions. The concepts of many value logic must be considered as abstractions, as tools that do not necessarily have any interpretation in the ordinary use of language or in physical observations. The third value in three value logic is not simply a temporarily undetermined value that must actually be 1 or 0 (true or false) but an additional value that is separate and distinct from the others. It is possible to construct systems using any number of values including infinite values.

Many value logical systems are of interest in connection with probability theory and may have application in developing decision elements for more efficient and flexible programming of many types of computing machines. It is inevitable that probability theory will be of increasing importance in the design of high order (high speed, large scale, etc.) digital computers.

The subject of many value logic is potentially extremely complex and has led to many misunderstandings and confusions. It is discussed here briefly for purposes of indicating the notions that may be developed and the possible extensions of decision elements on such a basis.

The term "distinguished value" is used to indicate that the field of a system described by a matrix contains only those theses having such a value. It is indicated by an asterisk. In n value logic where n is a positive integer it is possible to construct systems having as many as n-1 distinguished values.

For purposes of intuitive understanding it is convenient to present a matrix for three value logic using for the variables the values $0, \frac{1}{2}$ and 1. The following is a matrix for implication and negation in such

a system with one distinguished value:

	C	0	$\frac{1}{2}$	1	N
	0	1	1	1	1
	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
*	1	0	$\frac{1}{2}$	1	0

This matrix is not canonical for two distinguished values. There are 3^9 possible functors for two arguments, and 27 possible functors for one argument in a three value logical structure. As an example of the manner in which the functors are extended, it is interesting to consider the expression Apq in bi-value logic. This expression may be defined (using only implication) as CCpqq, and it may also be defined (using implication and negation) as CNpq. But in three value logic the matrix for CCpqq, symbolized as A_1, is:

	A_1	0	$\frac{1}{2}$	1
	0	0	1	1
	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
*	1	1	1	1

and the matrix for CNpq, symbolized as A_2, is:

	A_2	0	$\frac{1}{2}$	1
	0	0	$\frac{1}{2}$	1
	$\frac{1}{2}$	$\frac{1}{2}$	1	1
*	1	1	1	1

Thus, in the basic concept of a decision element A in bi-value logic, there is contained the potential three value decision elements A_1 and A_2.

Using two distinguished values, the following matrix for C_s and H may be constructed:

	C_s	1	2	3	H
	1	3	2	3	2
*	2	1	3	3	3
*	3	1	2	3	1

Any implicational thesis without negation satisfies this matrix. Using C_s and H, it is possible to define all functors for three value logic.

Three systems have been indicated, one bi-value system and two three value systems. There are some theses that appear in all three systems.

This procedure may be extended to construct n-value logics using additional fractional numbers to indicate the various values, although this symbolism becomes clumsy and it is more practical to use positive integers to symbolize the values.

Complete systems using many value logical functors have many interesting possibilities. From the technical point of view it is necessary to design devices that are capable of both input and output responses in terms of three or more forms of the electrical signals. For three value systems it is possible to consider using pulses with half the normal width or half the normal amplitude or perhaps negative pulses as the third value.

Analog and Digital Systems

Analog computers are devices in which quantities are represented and handled in continuums. Digital computers represent and handle the information in discrete units. In general, these systems have been used separately and machines have been designed to operate in accordance with one or the other of these principles. In many applications of computing machinery, particularly in control devices, the input information must be accepted in analog form, such as a shaft rotation, a lever motion, etc. At the present time there is a strong tendency to digitalize all such information and handle the computing problems in discrete unit systems, and then, if required by the application, retranslate the solutions into analog presentation.

It is quite possible that in many internal operations of digital equipment temporary handling of information in a form that partakes of the "lumped" characteristic of analog presentation will be useful. For example, in connection with any device that is capable of a multiplicity of stable states, it may be desirable to represent a number n as a single pulse that is n times as long or as high as, or has n times the integral of, some standard unit-digit pulse. In such a system the intelligence might be temporarily shifted between structures in analog form and redigitalized as required for purposes of maintaining accuracy.

This leads to one possible application for the many value logical functors. Decision elements based on these principles may well fit into the pattern of partial analog handling of intelligence in machines that are essentially digital.

REFERENCES

1. The notation placing the functors ahead of the variables was originated by Jan Łukasiewicz. (Aristotle's Syllogistic, 1951.)

2. The suggested notation for decision elements is derived from Stanisław Leśniewski's symbolism:

    (a) Grundzüge eines neuen System der Grundlagen der Mathematik. Fundamenta Mathematicae, vol. XIV, Warsaw, 1929.

    (b) Enleitende Bemerkungen zur Fortsetzung meiner Mitteilung u.d.T. "Grundzüge eines neuen System der Grundlagen der Mathematik," Collectanea Logica, vol. I, Warsaw, 1938.

    (c) Grundzüge eines neuen System der Grundlagen der Mathematik, §12. Collectanea Logica, vol. I, Warsaw, 1938.

3. Synthesis of Electronic Computing and Control Circuits, Harvard University Press, 1951, Chapt. 11.

THE REALIZATION OF A UNIVERSAL DECISION ELEMENT

TENNY LODE

The concept of the Decision Element is extremely powerful in the digital computation field. The entire mechanism of any digital computing machine may be considered as a complex decision device. The output is entirely a function of the input (and stored) data. Hence, a computer program may be reduced to a series of basic decisions and its mechanisms may be similarly reduced to a series of simple decision elements.

It is the purpose of this paper to illustrate certain elementary properties of systems of D.E.'s and to initially indicate means for the practical realization of such structures with a minimum of component parts.

The following is a list of the special terms used and their definitions:

Decision Element: A device producing a digital output as a function of a number of digital inputs.

Radix: The arithmetic base of the system for which a given D.E. is applicable.

Order: The number of inputs to the D.E.

Rank: The number of inputs of which the output is actually a function. (Rank is less than or equal to order.)

"X" Valence: The total number of separate combinations of input signals which will produce output "X". In a binary system (radix 2) "X" is understood to be 1.

Symmetric: A D.E. is symmetric when it remains invariant through an interchange of inputs. N order symmetry is symmetry of N inputs with respect to each other.

Similar: Two D.E.'s are similar when they may be made identical by interchange of inputs. (Two similar symmetric D.E.'s are identical.)

The present paper is essentially restricted to consideration of systems of D.E.'s of radix 2 and order 1 and 2. Systems of greater complexity will be treated subsequently.

The total number of D.E.'s in a given system is readily calculated. The number of possible combinations of input signals is R^n, where R is the radix and n the order. Hence the total number of D.E.'s is given by $R^{(R^n)}$. A few values of this function are given in Table I.

(Received Jan. 15, 1952)

THE REALIZATION OF A UNIVERSAL DECISION ELEMENT

Of the 16 D.E.'s of R-2, N-2 only the 8 marked * are of importance in computer design and the remaining 8 are trivial for the reasons indicated.

It is true that a large portion of the D.E.'s in a given system are of trivial interest to the computing machinery field. The order of magnitude is, however, staggering for any except the lowest order and radix.

Any D.E. of order 3 or more may be synthesized from a configuration of D.E.'s of lower order, and hence of order 2. For this reason and others the greatest interest in the computer field has been in systems of radix 2 (binary) and orders 1 and 2.

Reference is made to Tables II and III.

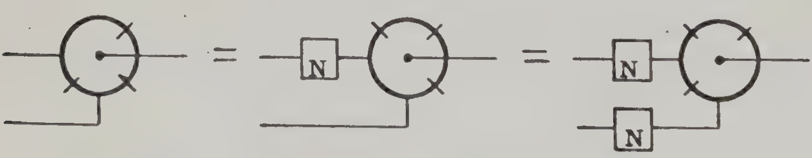
Of the 16 D.E.'s of R-2, N-2 only the 8 marked * are of importance in computer design and the remaining 8 are trivial for the reasons indicated.

Current practice in the industry is to provide special circuitry for each of the 8 significant D.E.'s. This is inefficient and undesirable for many reasons.

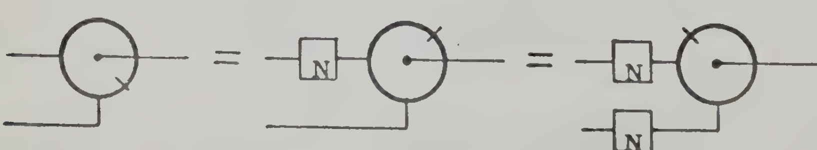
It is known that D.E.'s of the type 9 or 15 may, in appropriate configurations, be used for the synthesis of a complete system. Other D.E.'s, 2,3,5,8,12 or 14 exhibit similar properties when combined with one of the listed functors for one argument. Such a synthesizing process is illustrated in Table IV.

This embodiment of the concept of the universal computing element, while powerful in principle, leads to an excess of basic units for the design of any given machine. Thus other structures are considered. The following properties of D.E.'s are evident:

A) The four valence - 3 D.E.'s may be related by input negation:

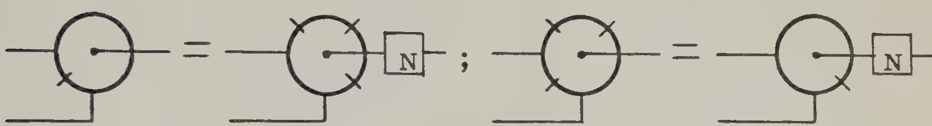


B) The four valence - 1 D.E.'s may be related by input negation:

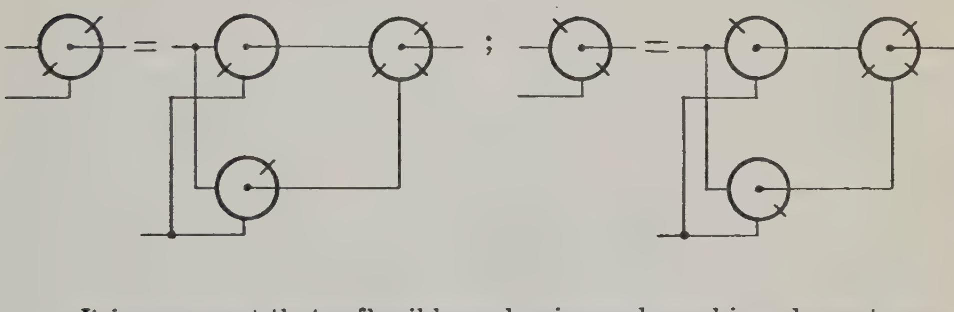


C) The two rank - 2, valence - 2 D.E.'s may be related by output negation:

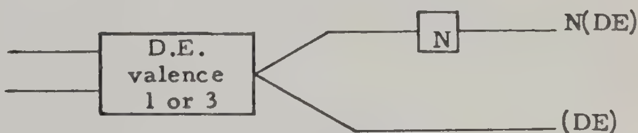
D) The sets of valence - 3 and valence - 1 may be related by output negation:



E) D.E.'s 7 and 10 (rank - 2, valence - 2) may be synthesized from D.E.'s of valence - 1 and 3:



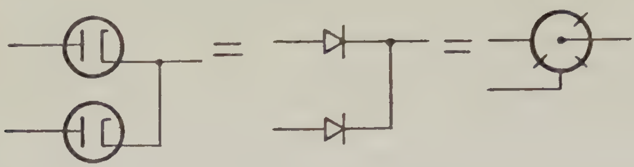
It is apparent that a flexible and universal machine element may be conceived by incorporating the negation functor. Specifically, an efficient form of universal element would be:



The availability of a negated output implies the availability of a negated input to all subsequent D.E.'s. A single universal D.E. of the type proposed is capable of operating as any odd valence D.E. The two significant valence - 2 D.E.'s are realized as the parallel output of 2 universal elements.

Machine-wise, D.E. number 8 (Non-exclusive or) is probably the most easily designed of the odd valence elements since it represents a

simple mixing circuit.



One physical circuit for such a universal element is shown below. Operation is non-synchronous and continuous in nature.

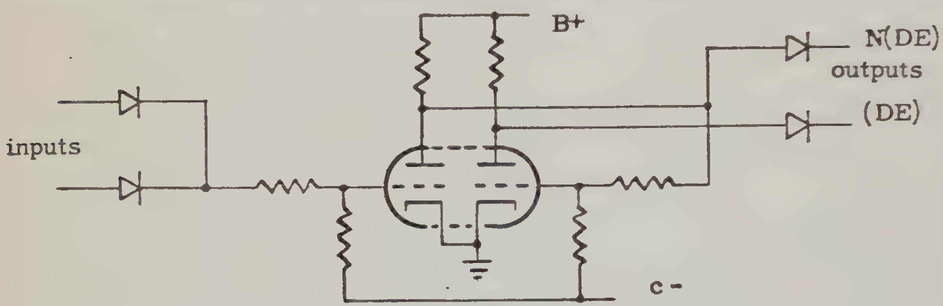


Table V illustrates the application of the universal element given above in the design of all significant D.E.'s of ranks 1 and 2.

Obviously it is possible to design such D.E.'s using many different techniques and types of component parts such as transistors, magnetic elements, etc. The use of negation will often result in component economy as well as simplified program structures by comparison with specialized circuitry.

Most important is the fact that specialized computer equipment may be truly a production line product with suitable inter-connections as required by the application. The real strength of the D.E. concept is recognized when it is appreciated that ANY digital computer (exclusive of input and output equipment) may be constructed exclusively of D.E.'s, and as indicated above this may be a single standardized basic component capable of synthesizing all D.E.'s.

It is of some interest to note that a similar universal D.E. is possible for three or more inputs and for a "variable functor" type of design.

Circuitry, programming systems and similar expansions of these concepts will be presented in future articles.

THE MINNESOTA ELECTRONICS CORPORATION
ST. PAUL, MINNESOTA

"N" (inputs)

"R" (radix)		1	2	3	4	5
	2	4	16	256	$6.55.10^{4}$	$4.29.10^{9}$
	3	27	$1.97.10^{4}$	$7.63.10^{12}$	$4.43.10^{38}$	$8.71.10^{115}$
	4	256	$4.29.10^{9}$	$3.40.10^{38}$	$1.34.10^{154}$	$3.23.10^{616}$
	5	3,125	$2.98.10^{17}$	$2.35.10^{87}$	$7.18.10^{436}$	$1.91.10^{2184}$

Table I

Values of $R^{(R^n)}$
(to three significant figures)

output

input		A	N	V	F
	0	0	1	1	0
	1	1	0	1	0

Table II

(The 4 Decision Elements of N-1, R-2)

	logical matrix	circuit symbol	rank	valence	symetry	related Order - 1 D.E. (inputs joined)	Order - 1 D.E. required for generation of complete system	logical symbols & Circuit Comments
1.)	0 0 0 0		0	0	Yes	F		"F" Trivial
• 2.)*	0 0 0 1		2	1	Yes	A	N	"K" "conjunction" ("and")
• 3.)*	0 0 1 0		2	1	No	F	N or V	inhibitor gate
4.)	0 0 1 1		1	2	No	A		Trivial
5.)	0 1 0 0		2	1	No	F	N or V	similar to No. 3
6.)	0 1 0 1		1	2	No	A		Trivial
• 7.)*	0 1 1 0		2	2	Yes	F		"R" "exclusive or"
• 8.)*	0 1 1 1		2	3	Yes	A	N	"A" "non-exclusive or"

Table III (part 1)

The 16 D.E.s of R-2, N-2

	logical matrix	circuit symbol	rank	valence	symetry	related order - 1 D.E. (inputs joined)	Order - 1 D.E. required for generation of complete system	logical symbols & comments
9.)*	1 0 0 0		2	1	Yes	N	self sufficient	"S"
10.)*	1 0 0 1		2	2	Yes	V		E "equivalence"
11.)	1 0 1 0		1	2	No	N		Trivial
12.)*	1 0 1 1		2	3	No	V	N or F	an uncommon element
13.)	1 1 0 0		1	2	No	N		Trivial
14.)	1 1 0 1		2	3	No	V	N or F	similar to No. 12
15.)*	1 1 1 0		2	3	Yes	N	self sufficient	"D"
16.)	1 1 1 1		0	4	Yes	V		"V" Oscillator - Clock

Table III (part 2)

The 16 D.E.s of R-2, N-2

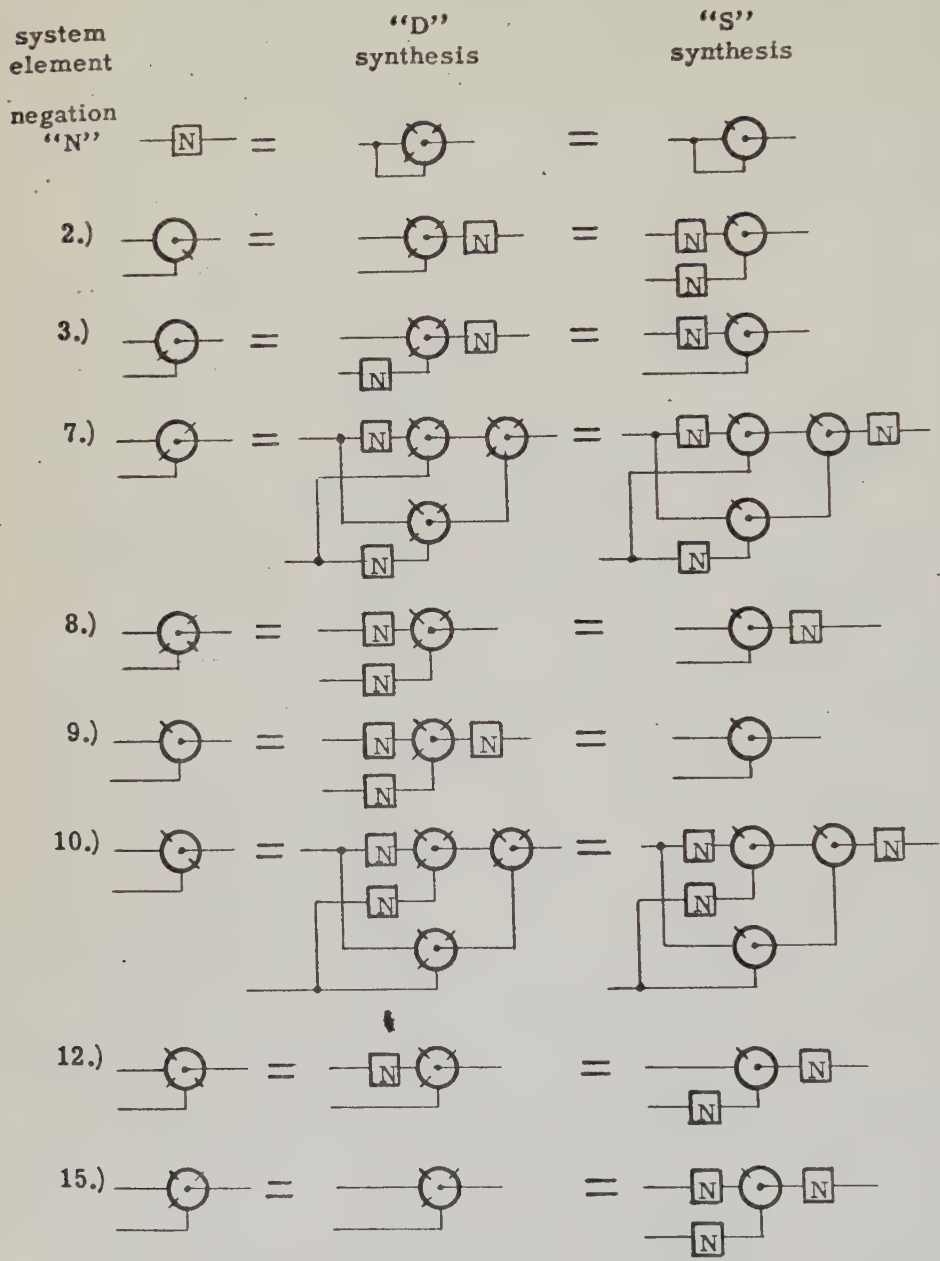


Table IV

System synthesis from elements "D"$^{(15)}$ and "S"$^{(9)}$

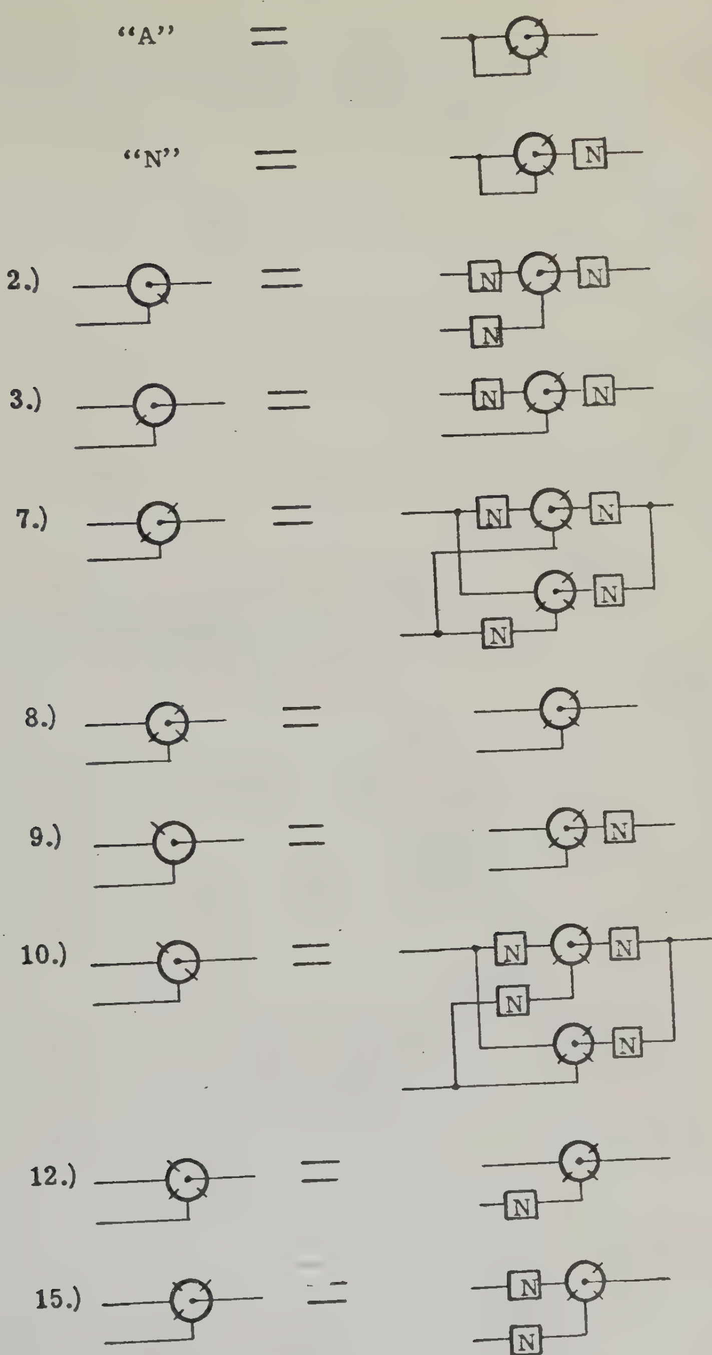


Table V
System synthesis from preferred universal D.E.

AXIOMATIZATION OF A PARTIAL SYSTEM OF THREE-VALUE CALCULUS OF PROPOSITIONS

BOLESŁAW SOBOCIŃSKI

The purpose of this paper is to present a partial system of the calculus of propositions. This system is based on the primitive terms "implication" and "negation", and is defined by the following matrix.[1]

	C	0	1	2	N
	0	1	1	1	1
*	1	0	1	0	0
*	2	0	1	2	2

in which "0" represents a false value and "1" and "2" - the true values. This system possesses the peculiar property, as I have shown in the past, that no thesis in which some variable appears only once will satisfy the given matrix.[2] I shall prove that if we adopt the rules of procedure ordinarily used in the calculus of propositions, i.e., the rule of substitution and the rule of detachment adjusted to the implication, then any thesis which satisfies this matrix is a consequence of the following five independent axioms:

1 CCpqCCqrCpr
2 CpCCpqq
3 CCpCpqCpq
4 CpCqCNqp
5 CCNpNqCqp

This investigation has interest not only because it gives an axiomatization of this partial system, but also because it shows that it is possible to construct a system of "material implication" which is finite, sufficiently rich and contains no thesis (with the exception of four) which is "paradoxical", in the meaning of the system of "strict implication".[3] Clearly, this system, having a finite adequate matrix, is not a system of strict implication and its primitive term "C" cannot be interpreted as strict implication. Nevertheless, this system can be useful as a system of "material implication" without almost all "paradoxical" theses.

In the following paragraphs I shall successively derive from the adopted axiomatic such theses as I require, give a proof of the axiomatization and show that each adopted axiom is independent from the others. The discussion about the relations of this system to some others is given toward the end.

(Received November 1, 1951)

PARTIAL SYSTEM OF THREE-VALUE CALCULUS

§1. Using the rules of procedure we can prove from the adopted axioms:

1 CCpqCCqrCpr
2 CpCCpqq
3 CCpCpqCpq
4 CpCqCNqp
5 CCNpNqCqp

the following theses:

1p/Cpq,q/CCqrCpr,r/s * C1 - 6
6 CCCCqrCprsCCpqs
6q/Cqr,r/Csr,s/CCsqCpCsr * C6p/s,s/CpCsr - 7
7 CCpCqrCCsqCpCsr
6s/CCCprsCCqrs * C1p/Cqr,q/Cpr,r/s - 8
*8 CCpqCCCprsCCqrs
7p/Cpq,q/CCprs,r/CCqrs,s/t * C8 - 9
9 CCtCCprsCCpqCtCCqrs
9p/q,q/r,r/s,s/Cps,t/Cpq * C1r/s - 10
10 CCqrCCpqCCrsCps
7p/q,q/Cqr,s/p * C2p/q,q/r - 11
*11 CCpCqrCqCpr
11p/Cpq,q/Cqr,r/Cpr * C1 - 12
*12 CCqrCCpqCpr
12p/s,q/CpCqr,r/CqCpr * C11 - 13
13 CCsCpCqrCsCqCpr
1p/CsCpCqr,q/CsCqCpr,r/CqCsCpr * C13 - C11p/s,r/Cpr - 14
14 CCsCpCqrCqCsCpr
1p/CsCpCqr,q/CqCsCpr,r/CqCpCsr * C14 - C13p/s,q/p,s/q - 15
15 CCsCpCqrCqCpCsr
1p/CsCpCqr,q/CqCpCsr,r/CpCqCsr * C15 - C11p/q,q/p,r/Csr - 16
16 CCsCpCqrCpCqCsr
12p/s,q/Cpq,r/CCqrCpr * C1 - 17
17 CCsCpqCsCCqrCpr
13p/s,q/Cqr,r/Cpr,s/CsCpq * C17 - 18
*18 CCsCpqCCqrCsCpr
12p/t,q/CsCpr,r/CCqrCsCpr * C18 - 19
19 CCtCsCpqCtCCqrCsCpr
13p/t,q/Cqr,r/CsCpr,s/CtCsCpq * C19 - 20
20 CCtCsCpqCCqrCtCsCpr
18p/Csq,q/CpCsr,r/CCtpCtCsr,s/CpCqr * C7 - C12p/t,q/p,r/Csr - 21
21 CCpCqrCCsqCCtpCtCsr
20p/Ctp,q/CtCsr,r/CCrvCtCsv,s/Csq,t/CpCqr * C21 - C18p/s,q/r,r/v,s/t - 22
22 CCpCqrCCsqCCtpCCrvCtCsv
1p/CpCsq,q/CsCpq,r/CCqrCsCpr * C11q/s,r/q - C18 - 23
23 CCpCsqCCqrCsCpr
13p/Cpq,q/p,s/Cqr * C12 - 24
*24 CCqrCpCCpqr
23p/Cqr,q/CCpqr,r/s,s/p * C24 - 25

*25 CCCCpqrsCpCCqrs
15p/CCprs,q/Cqr,r/s,s/Cpq * C8 - 26

*26 CCqrCCCprsCCpqs
17p/CCprs,q/CCqrs,r/t,s/Cpq * C8 - 27

27 CCpqCCCCqrstCCCprst
18p/Cpq,q/CpCpr,r/Cpr,s/CpCqr * C7s/p - C3q/r - 28

28 CCpCqrCCpqCpr
18p/Csp,q/CsCqr,r/CCsqCsr,s/CpCqr * C12p/s,q/p,r/Cqr - C28p/s - 29

*29 CCpCqrCCspCCsqCsr
1p/CCpqCCCprsCCqrs,q/CCCpqCCprsCCpqCCqrs,r/CCCpqCCprsCCqrCCpqs * C28p/Cpq,q/CCprs,r/CCqrs - C13p/Cpq,q/Cqr,r/s,s/CCpqCCprs - C8 - 30

*30 CCCpqCCprsCCqrCCpqs
1p/CCpqCCprs,q/CCprCCpqs,r/CCrqCCprs * C11p/Cpq,q/Cpr,r/s - C30q/r,r/q - 31

*31 CCCpqCCprsCCrqCCprs
20p/Csq,q/CsCrv,r/CCsrCsv,s/Csp,t/CpCqCrv * C29r/Crv - C28p/s,q/r,r/v - 32

32 CCpCqCrvCCspCCsqCCsrCsv
1p/CpCqCrv,q/CCspCCsqCCsrCsv,r/CCtCspCCtCsqCCtCsrCtCsv * C32 - C32p/Csp,q/Csq,r/Csr,s/t,v/Csv - 33

33 CCpCqCrvCCtCspCCtCsqCCtCsrCtCsv
3q/CNpp * C4q/p - 34

*34 CpCNpp
11q/Np,r/p * C34 - 35

*35 CNpCpp
1p/NNp,q/CNpNp,r/Cpp * C35p/Np - C5q/p - 36

36 CNNpCpp
11p/NNp,q/p,r/p * C36 - 37

37 CpCNNpp
1p/Np,q/CNNNpNp,r/CpNNp * C37p/Np - C5p/NNp,q/p - 38

38 CNpCpNNp
1p/NNp,q/CNpNNNp,r/CNNpp * C38p/Np - C5q/NNp - 39

39 CNNpCNNpp
3p/NNp,q/p * C39 - 40

*40 CNNpp
5p/NNp,q/p * C40p/Np - 41

*41 CpNNp
1q/NNp,r/CNpNp * C41 - C35p/Np - 42

42 CpCNpNp
11q/Np,r/Np * C42 - 43

*43 CNpCpNp
1q/CNpNp,r/Cpp * C42 - C5q/p - 44

*44 CpCpp
3q/p * C44 - 45

*45 Cpp
18q/p,r/q,s/p * C44 - 46

*46 CCpqCpCpq
23p/Cpq,q/Cpq,s/p * C46 - 47

*47 CCCpqrCpCCpqr
7p/q,r/q,s/p * C44p/q - 48

*48 CCpqCqCpq
23p/Cpq,q/Cpq,s/q * C48 - 49

*49 CCCpqrCqCCpqr
1p/Cpq,q/CqCpq,r/CpCqq * C48 - C11p/q,q/p,r/q - 50

50 CCpqCpCqq
18q/Crr,r/CCqrCqr,s/Cpr * C50q/r - C12p/q,q/r - 51

51 CCprCpCCqrCqr
14q/Cqr,r/Cqr,s/Cpr * C51 - 52

52 CCqrCCprCpCqr
23p/Cqr,q/CpCqr,r/s,s/Cpr * C52 - 53

*53 CCCpCqrsCCprCCqrs
10p/Cpq,q/CCqrCpr,r/CqCCqrCpr,s/CCqrCqCpr * C47p/q,
q/r,r/Cpr - C1 - C11p/q,q/Cqr,r/Cpr - 54

54 CCpqCCqrCqCpr
23p/CCpqr,q/CCpqr,r/s,s/q * C49 - 55

55 CCCCpqrsCqCCCpqrs
10p/CpCqr,q/CpCqr,r/CpCpCqr,s/CqCpCpr * C46q/Cqr -
C45p/CpCqr - C15s/p - 56

56 CCpCqrCqCpCpr
10p/CpCqr,q/CqCpr,r/CqCqCpr,s/CqCpCqr * C46p/q,q/Cpr
- C11 - C13p/q,q/p,s/q - 57

57 CCpCqrCqCpCqr
8p/NNp,q/p,r/Nq,s/CqNp * C40 - C5p/Np - 58

*58 CCpNqCqNp
26p/Np,r/NNq,s/CNqp * C41p/q - C5q/Nq - 59

*59 CCNpqCNqp
1q/NNp,r/q * C41 - 60

*60 CCNNpqCpq
1p/NNp,q/p,r/q * C40 - 61

*61 CCpqCNNpq
12q/NNq,r/q * C40p/q - 62

*62 CCpNNqCpq
12r/NNq * C41p/q - 63

*63 CCpqCpNNq
1p/Cpq,q/CNNpq,r/CNqNp * C61 - C59p/Np - 64

*64 CCpqCNqNp
8p/CNpNq,q/Cqp * C5 - C65

65 CCCCNpNqrsCCCqprs
1p/CCNqrCNrq,q/CCpCNqrCpCNrq,r/CCpCNqrCNrCpq *
C12q/CNqr,r/CNrq - C13q/Nr,r/q,s/CpCNqr - C59p/q,q/r
- 66

66 CCpCNqrCNrCpq
1p/CCqrCNrNq,q/CCpCqrCpCNrNq,r/CCpCqrCNrCpNq *
C12q/Cqr,r/CNrNq - C13q/Nr,r/Nq,s/CpCqr - C64p/q,q/r - 67

67 CCpCqrCNrCpNq
1p/CqCpr,q/CpCqr,r/CNrCpNq * C11p/q,q/p - C67 - 68

*68 CCqCprCNrCpNq
1p/CNqCpr,q/CpCNqr,r/CNrCpq * C11p/Nq,q/p - C66 - 69

69 CCNqCprCNrCpq
1p/Np,q/CCNpqq,r/CNqNCNpq * C2p/Np - C64p/CNpq - 70

70 CNpCNqNCNpq
3p/Np,q/NCNpp * C70q/p - 71

71 CNpNCNpp
5q/CNpp * C71 - 72

*72 CCNppp
8p/NNp,q/p,r/Np,s/Np * C40 - C72p/Np - 73

*73 CCpNpNp
22p/CNpNq,q/CNqp,r/CNpp,s/CNqp,t/Cqp,v/p * C1p/Np,q/Nq, r/p - C45p/CNqp - C64p/q,q/p - C72 - 74

74 CCqpCCNqpp
18p/Cqr,q/Cpr,r/CCNprr,s/Cpq * C1 - C74p/r,q/p - 75

75 CCpqCCqrCCNprr
15p/Cqr,q/CNpr,s/Cpq * C75 - 76

*76 CCNprCCqrCCpqr
29p/CNpr,q/Cqr,r/CCpqr * C76 - 77

77 CCsCNprCCsCqrCsCCpqr
1p/CNpr,q/CCqrCCpqr,r/CCsCqrCsCCpqr * C76 - C12p/s, q/Cqr,r/CCpqr - 78

78 CCNprCCsCqrCsCCpqr
7p/q,q/Nq,r/Nq,s/p * C42p/q - 79

*79 CCpNqCqCpNq
7p/Nq,r/q,s/p * C35p/q - 80

80 CCpqCNqCpq
23p/Np,q/Np,r/q,s/p * C43 - 81

*81 CCNpqCpCNpq
18q/p,r/q,s/Np * C35 - 82

*82 CCpqCNpCpq
23p/CNpq,q/CNpq,s/p * C81 - 83

*83 CCCNpqrCpCCNpqr
23p/Cpq,q/Cpq,s/Np * C82 - 84

84 CCCpqrCNpCCpqr
23p/CCpqr,q/CCpqr,r/s,s/Np * C84 - 85

85 CCCCpqrsCNpCCCpqrs
23q/p,r/q,s/NNp * C37 - 86

86 CCpqCNNpCpq
1p/CCqrCNrCqr,q/CCpCqrCpCNrCqr,r/CCpCqrCNrCpCqr * C12q/Cqr,r/CNrCqr - C13q/Nr,r/Cqr,s/CpCqr - C80p/q,q/r - 87

87 CCpCqrCNrCpCqr
10p/CpCqr,q/CNrCpCqr,r/CNCqrCpr,s/CNCqrCNCqrCpr * C69q/r,r/Cqr - C87 - C46p/NCqr,q/Cpr - 88

88 CCpCqrCNCqrCNCqrCpr
10p/CpCqr,q/CNrCpCqr,r/CNCqrCpr,s/CNCprCNCqrCpr * C69q/r,r/Cqr - C87 - C80p/NCqr,q/Cpr - 89

89 CCpCqrCNCprCNCqrCpr
20q/CNrr,s/q,t/CNrCpCqr * C15s/Nr - C72p/r - 90

90 CCNrCpCqrCqCpr
10p/CNCqrCpr,q/CNrCpCqr,r/CqCpr,s/CpCqCqr * C90 - C69q/C qr - C56p/q,q/p - 91

91 CCNCqrCprCpCqCqr
10p/CNCqrCpr,q/CpCqCqr,r/CNCqrCqNp,s/CNCqNpCqr * C68p/q, q/p,r/Cqr - C91 - C59p/Cqr,q/CqNp - 92

92 CCNCqrCprCNCqNpCqr
1p/CpCqCNqp,q/CpCNqCqp,r/CNCqpCpq * C13p/q,q/Nq,r/p,s/p - C66r/Cqp - C4 - 93

*93 CNCqpCpq
67p/Cpq,r/Cpq * C48 - 94

*94 CNCpqCCpqNq
66p/Cpq,r/Cpq * C80 - 95

*95 CNCpqCCpqq
1p/CCpqq,q/CCpqq,r/CNqNCpq * C45p/CCpqq - C64p/Cpq - 96

96 CCCpqqCNqNCpq
78p/Cpq,q/Nq,r/NCpq,s/CCpqq * C45p/NCpq - C96 - 97

*97 CCCpqqCCCpqNqNCpq
78p/q,q/r,r/Cqq,s/CrCqq * C35p/q - C45p/CrCqq - 98

98 CCrCqqCCqrCqq
13p/Cqr,r/q,s/CrCqq * C98 - 99

*99 CCrCqqCqCCqrq
78p/Np,r/Cpp,s/CqCpp * C36 - C45p/CqCpp - 100

100 CCqCppCCNpqCpp
13p/CNpq,q/p,r/p,s/CqCpp * C100 - 101

*101 CCqCppCpCCNpqp
18p/Cqr,q/CCpqr,r/s,s/CNpr * C76 - 102

*102 CCCCpqrsCCNprCCqrs
27p/Cpq,q/CNqNp,t/CNqCCNprs * C64 - C25p/Nq,q/Np - 103

*103 CCCCpqrsCNqCCNprs
23p/q,q/CNpq,s/p * C4p/q,q/p - 104

*104 CCCNpqrCpCqr
27q/NNp,r/q,s/r,t/CNpCqr * C41 - C104p/Np - 105

*105 CCCpqrCNpCqr
13p/Np,s/CCpqr * C105 - 106

*106 CCCpqrCqCNpr
18p/q,q/CNpp,r/p,s/CCpqp * C106r/p - C72 - 107

107 CCCpqpCqp
1p/CCCpqpCqp,q/CCrCCpqpCrCqp,r/CCrCCpqpCqCrp * C12p/r, q/CCpqp,r/Cqp - C13p/r,r/p,s/CrCCpqp - C107 - 108

*108 CCrCCpqpCqCrp
14s/CCNpqr * C104 - 109

109 CqCCCNpqrCpr
78q/CCNpqp,r/Cpp,s/q * C35 - C109r/p - 110

110 CqCCpCCNpqpCpp
11p/q,q/CpCCNpqp,r/Cpp * C110 - 111

*111 CCpCCNpqpCqCpp
22p/CNpCNrNq,q/CNpNr,r/CNpNq,s/Crp,t/CNpCNrNq,v/Cqp * C28p/Np,q/Nr,r/Nq - C64p/r,q/p - C45p/CNpCNrNq - C5 - 112

112 CCNpCNrNqCCrpCqp
26p/Np,q/Cqr,r/CNrNq,s/CCrpCqp * C64p/q,q/r - C112 - 113

113 CCNpCqrCCrpCqp
1p/Cqr,q/CNqCqr,r/CCpNqCpCqr * C82p/q,q/r - C12q/Nq,r/Cqr - 114

114 CCqrCCpNqCpCqr
23p/Cqr,q/CpCqr,r/s,s/CpNq * C114 - 115

*115 CCCpCqrsCCpNqCCqrs
8p/CqCpr,q/CpCqr,r/s,s/CCqNpCCprs * C11p/q,q/p - C115p/q, q/p - 116

*116 CCCpCqrsCCqNpCCprs
22p/Cpq,q/Cqr,r/CqCpr,s/NCrq,t/NCqp,v/CCsqCCspCsr * C54 - C93q/r,p/q - C93 - C29p/q,q/p - 117

117 CNCqpCNCrqCCsqCCspCsr
22p/Csr,q/NCsr,r/Csr,s/NCsr,t/NCrs,v/CCqsCqr * C34p/Csr - C45p/NCsr - C93p/s,q/r - C12p/q,q/s - 118

118 CNCrsCNCsrCCqsCqr
1p/CpCqr,q/CqCpCpr,r/CNCprCpNq * C56 - C68r/Cpr - 119

119 CCpCqrCNCprCpNq
1p/CpCqr,q/CqCpCqr,r/CNCqrCpNq * C57 - C68r/Cqr - 120

120 CCpCqrCNCqrCpNq
77p/Cqr,q/NCpr,r/CpNq,s/CpCqr * C120 - C119 - 121

121 CCpCqrCCCqrNCprCpNq
11p/CpCqr,q/CCqrNCpr,r/CpNq * C121 - 122

122 CCCqrNCprCCpCqrCpNq
49p/Cqr,q/NCpr,r/CCpCqrCpNq * C122 - 123

123 CNCprCCCqrNCprCCpCqrCpNq
77p/Np,r/CpNq,s/CpNq * C86q/Nq - C79 - 124

124 CCpNqCCNpqCpNq
7p/CpNq,q/CNpq,r/CpNq,s/NCqNp * C124 - C93p/Np - 125

125 CCpNqCNCqNpCpNq
18p/CpCqr,q/CpNq,r/CNCqNpCpNq,s/CCqrNCpr * C122 - C125 - 126

126 CCCqrNCprCCpCqrCNCqNpCpNq
14p/CpCqr,q/NCqNp,r/CpNq,s/CCqrNCpr * C126 - 127

127 CNCqNpCCCqrNCprCCpCqrCpNq
76p/Cpr,q/NCqNp,r/CCCqrNCprCCpCqrCpNq * C123 - C127 - 128

128 CCCprNCqNpCCCqrNCprCCpCqrCpNq
77p/Cqr,q/NCpr,r/CNCqrCpr,s/CpCqr * C88 - C89 - 129

129 CCpCqrCCCqrNCprCNCqrCpr
15p/CCqrNCpr,q/NCqr,r/Cpr,s/CpCqr * C129 - 130

130 CNCqrCCCqrNCprCCpCqrCpr
18p/CCqrNCpr,q/CNCqrCpr,r/CNCqNpCqr,s/CpCqr * C129 - C92 - 131

131 CCpCqrCCCqrNCprCNCqNpCqr
15p/CCqrNCpr,q/NCqNp,r/Cqr,s/CpCqr * C131 - 132

132 CNCqNpCCCqrNCprCCpCqrCqr
77p/q,q/p,r/Cpq,s/Cpq * C80 - C46 - 133

133 CCpqCCqp Cpq
10p/Cpq,q/CCqpCpq,r/CCrCqpCrCpq,s/CCrCqpCpCrq * C14p/r, q/Cqp,r/Cpq - C133 - C13p/r,q/p,r/q,s/CrCqp - 134

134 CCpqCCrCqpCpCrq
77p/q,q/s,r/CCqrs,s/CCqrs * C84p/q,q/r,r/s - C48p/Cqr,q/s - 135

135 CCCqrsCCqsCCqrs
20p/r,q/CCqrs,r/CCqsCCqrs,s/CCqrCsr,t/Crs * C134p/r, q/s,r/Cqr - C135 - 136

136 CCrsCCCqrCsrCrCCqsCCqrs
22p/Crs,q/CCqrCsr,r/CrCCqsCCqrs,s/CCqrCsr,t/NCsr,v/CCqs CCqrCrs * C136 - C45p/CCqrCsr - C93p/r,q/s - C16p/Cqs,q/Cq r,r/s,s/r - 137

137 CNCsrCCCqrCsrCCqsCCqrCrs
16p/CCqrCsr,q/Cqs,r/CCqrCrs,s/NCsr * C137 - 138

138 CCCqrCsrCCqsCNCsrCCqrCrs
18q/CNpr,r/q,s/CNpr * C81q/r - 139

139 CCCNprqCCNprCpq
23p/CCNprq,q/Cpq,r/s,s/CNpr * C139 - 140

140 CCCpqsCCNprCCCNprqs
1p/CCpqs,q/CCNprCCCNprqs,r/CCtCNprCtCCCNprqs * C140 -
C12p/t,q/CNpr,r/CCCNprqs - 141

141 CCCpqsCCtCNprCtCCCNprqs
28p/CCpqr,q/CqCNpr,r/CqCCCNprqr * C141s/r,t/q - C106 -
142

142 CCCpqrCqCCCNprqr
14p/q,q/CCNprq,s/CCpqr * C142 - 143

143 CCCNprqCCCpqrCqr
1p/CCNNqrNp,q/CCCNqNprCNpr,r/CCCpqrCNpr * C143p/Nq,
q/Np - C65p/q,q/p,s/CNpr - 144

144 CCCNNqrNpCCCpqrCNpr
27p/q,q/NNq,s/Np,t/CCCpqrCNpr * C41p/q - C144 - 145

145 CCCqrNpCCCpqrCNpr
1p/CCqrNp,q/CCCpqrCNpr,r/CNCNprCCCpqrCNpr * C145 -
C80p/CCpqr,q/CNpr - 146

146 CCCqrNpCNCNprCCCpqrCNpr
1p/CCqrNp,q/CCCpqrCNpr,r/CqCCCpqrCNpr * C145 - C55s/CNpr
- 147

147 CCCqrNpCqCCCpqrCNpr
77p/CNpr,r/CCCpqrCNpr,s/CCqrNp * C146 - C147 - 148

148 CCCqrNpCCCNprqCCCpqrCNpr
1p/CCNprq,q/CCCpqrCqr,r/CNCqrCCCpqrCqr * C143 - C80p/CCpq
r,q/Cqr - 149

149 CCCNprqCNCqrCCCpqrCqr
1p/CCNprq,q/CCCpqrCqr,r/CNpCCCpqrCqr * C143 - C85s/Cqr -
150

150 CCCNprqCNpCCCpqrCqr
77p/Cqr,q/Np,r/CCCpqrCqr,s/CCNprq * C149 - C150 - 151

151 CCCNprqCCCqrNpCCCpqrCqr
29p/CCqrNp,q/CCNprq,r/CCCpqrCNpr * C148 - 152

152 CCsCCqrNpCCsCCNprqCsCCCpqrCNpr
22p/CNsCCqrNp,q/CNsCCNprq,r/CNsCCCpqrCNpr,s/CNqCCNprs,
t/CpCCqrs,v/CCCNprsCCCpqrs * C152s/Ns - C69p/CNpr,r/s -
C68p/Cqr,q/p,r/s - C113p/s,q/CCpqr,r/CNpr - 153

153 CCpCCqrsCCNqCCNprsCCCNprsCCCpqrs
14p/CNqCCNprs,q/CCNprs,r/CCCpqrs,s/CpCCqrs *C153 - 154

154 CCCNprsCCpCCqrsCCNqCCNprsCCCpqrs
29p/CCNprq,q/CCqrNp,r/CCCpqrCqr * C151 - 155

155 CCsCCNprqCCsCCqrNpCsCCCpqrCqr
22p/CNsCCNprq,q/CNsCCqrNp,r/CNsCCCpqrCqr,s/CpCCqrs,
t/CNqCCNprs,v/CNCqrCCCpqrs * C155s/Ns - C68p/Cqr,q/p,r/s -
C69p/CNpr,r/s - C69p/CCpqr,q/s,r/Cqr - 156

156 CCNqCCNprsCCpCCqrsCNCqrCCCpqrs
15p/CpCCqrs,q/NCqr,r/CCCpqrs,s/CNqCCNprs * C156 - 157

157 CNCqrCCpCCqrsCCNqCCNprsCCCpqrs
76p/Cqr,q/CCNprs,r/CCpCCqrsCCNqCCNprsCCCpqrs * C157
- C154 - 158

158 CCCqrCCNprsCCpCCqrsCCNqCCNprsCCCpqrs
1p/CCNprCCqrs,q/CCqrCCNprs,r/CCpCCqrsCCNqCCNprsCCCpqrs
* C11p/CNpr,q/Cqr,r/s - C158 - 159

*159 CCCNprCCqrsCCpCCqrsCCNqCCNprsCCCpqrs
29p/CCprCNqNp,q/CCqrNCpr,r/CCpCqrCpNq * C128 - 160
160 CCsCCprNCqNpCCsCCqrNCprCsCCpCqrCpNq
22p/CNsCCprNCqNp,q/CNsCCqrNCpr,r/CNsCCpCqrCpNq,s/CCpr CCqrs,t/CCqNpCCprs,v/CNCpNqCCpCqrs * C160s/Ns - C68p/Cqr, q/Cpr,r/s - C68p/Cpr,q/CqNp,r/s - C69p/CpCqr,q/s,r/CpNq - 161
161 CCCqNpCCprsCCCprCCqrsCNCpNqCCpCqrs
14p/CCprCCqrs,q/NCpNq,r/CCpCqrs,s/CCqNpCCprs * C161 - 162
162 CNCpNqCCCqNpCCprsCCCprCCqrsCCpCqrs
29p/NCqr,q/CCqrNCpr,r/CCpCqrCpr * C130 - 163
163 CCsNCqrCCsCCqrNCprCsCCpCqrCpr
22p/CNsNCqr,q/CNsCCqrNCpr,r/CNsCCpCqrCpr,s/CCprCCqrs, t/CCqrs,v/CCCprsCCpCqrs * C163s/Ns - C68p/Cqr,q/Cpr,r/s - C64p/Cqr,q/s - C113p/s,q/CpCqr,r/Cpr - 164
164 CCCqrsCCCprCCqrsCCCprsCCpCqrs
13p/CCprCCqrs,q/CCprs,r/CCpCqrs,s/CCqrs * C164 - 165
165 CCCqrsCCCprsCCCprCCqrsCCpCqrs
1p/NCqNp,q/CCCqrNCprCCpCqrCqr,r/CCsCCqrNCprCsCCpCqr Cqr * C132 - C12p/s,q/CCqrNCpr,r/CCpCqrCqr - 166
166 CNCqNpCCsCCqrNCprCsCCpCqrCqr
22p/NCqNp,q/CNsCCqrNCpr,r/CNsCCpCqrCqr,s/CCprCCqrs, t/NCqNp,v/CCCqrsCCpCqrs * C166s/Ns - C68p/Cqr,q/Cpr, r/s - C45p/NCqNp - C113p/s,q/CpCqr,r/Cqr - 167
167 CNCqNpCCCprCCqrsCCCqrsCCpCqrs
14p/CCprCCqrs,q/CCqrs,r/CCpCqrs,s/NCqNp * C167 - 168
168 CCCqrsCNCqNpCCCprCCqrsCCpCqrs
77p/CqNp,q/CCprs,r/CCCprCCqrsCCpCqrs,s/CCqrs * C168 - C165 - 169
169 CCCqrsCCCqNpCCprsCCCprCCqrsCCpCqrs
76p/CpNq,q/CCqrs,r/CCCqNpCCprsCCCprCCqrsCCpCqrs * C162 - C169 - 170
*170 CCCpNqCCqrsCCCqNpCCprsCCCprCCqrsCCpCqrs
29p/NCqp,q/NCrq,r/CCsqCCspCsr,s/t * C117 - 171
171 CCtNCqpCCtNCrqCtCCsqCCspCsr
22p/CNtNCqp,q/CNtNCrq,r/CNtCCsqCCspCsr,s/CCrqt,t/CCqpt, v/CCsqCNtCCspCsr * C171t/Nt - C64p/Crq,q/t - C64p/Cqp,q/t - C11p/Nt,q/Csq,r/CCspCsr - 172
172 CCCqptCCCrqtCCsqCNtCCspCsr
14p/CCrqt,q/Csq,r/CNtCCspCsr,s/CCqpt * C172 - 173
173 CCsqCCCqptCCCrqtCNtCCspCsr
20p/CCrqt,q/CNtCCspCsr,r/CCCsrtCCspt,s/CCqpt,t/Csq * C173 - C113p/t,q/Csp,r/Csr - 174
174 CCsqCCCqptCCCrqtCCCsrtCCspt
1p/CNCrsCNCsrCCqsCqr,q/CCpNCrsCpCNCsrCCqsCqr,r/CCp NCrsCNCsrCpCCqsCqr * C12q/NCrs,r/CNCsrCCqsCqr - C13q/NC sr,r/CCqsCqr,s/CpNCrs - C118 - 175
175 CCpNCrsCNCsrCpCCqsCqr
1p/CpNCrs,q/CNCsrCpCCqsCqr,r/CCtNCsrCtCpCCqsCqr * C175 - C12p/t,q/NCsr,r/CpCCqsCqr - 176
176 CCpNCrsCCtNCsrCtCpCCqsCqr
22p/CpNCrs,q/CNtNCsr,r/CNtCpCCqsCqr,s/CCsrt,t/CCrsNp,

v/CCqsCNtCpCqr * C176t/Nt - C64p/Csr,q/t - C58p/Crs,q/p - C14q/Cqs,r/Cqr,s/t - 177

177 CCCrsNpCCCsrtCCqsCNtCpCqr
20p/Cqs,q/CNtCpCqr,r/CCCqrtCpt,s/CCsrt,t/CCrsNp * C177 - C113p/t,q/p,r/Cqr - 178

178 CCCrsNpCCCsrtCCqsCCCqrtCpt
15p/CCsrt,q/Cqs,r/CCCqrtCpt,s/CCrsNp * C178 - 179

179 CCqsCCCsrtCCCrsNpCCCqrtCpt
26p/Cqr,q/NCrs,r/Csr,s/CCqsCNCsrCCqrCrs * C93p/s,q/r - C138 - 180

180 CCqrNCrsCCqsCNCsrCCqrCrs
18p/Cqs,q/CNCsrCCqrCrs,r/CCtNCsrCtCCqrCrs,s/CCqrNCrs * C180 - C12p/t,q/NCsr,r/CCqrCrs - 181

181 CCCqrNCrsCCqsCCtNCsrCtCCqrCrs
16p/Cqs,q/CtNCsr,r/CtCCqrCrs,s/CCqrNCrs * C181 - 182

182 CCqsCCtNCsrCCCqrNCrsCtCCqrCrs
7p/Cqs,q/CNtNCsr,r/CCCqrNCrsCNtCCqrCrs,s/CCsrt * C182t/Nt - C64p/Csr,q/t - 183

183 CCqsCCCsrtCCCqrNCrsCNtCCqrCrs
20p/CCqrNCrs,q/CNtCCqrCrs,r/CCCrstCCqrt,s/CCsrt,t/Cqs * C183 - C113p/t,q/Cqr,r/Crs - 184

184 CCqsCCCsrtCCCqrNCrsCCCrstCCqrt
33p/CCCrsNCqrCCCqrtCCqrt,q/CCCqrNCrsCCCrstCCqrt,r/CCCrstCCCqrtCCqrt,s/CCsrt,t/Cqs,v/CCCrsCCqrtCCqrt * C170p/Crs, q/Cqr,r/t,s/CCqrt - C179p/Cqr - C184 - C174p/r,q/s,s/q - 185

185 CCqsCCCsrtCCCrsCCqrtCCqrt
28p/Cqs,q/CCsrt,r/CCCrsCCqrtCCqrt * C185 - 186

186 CCCqsCCsrtCCqsCCCrsCCqrtCCqrt
10p/CCsrCCqst,q/CCqsCCsrt,r/CCqsCCCrsCCqrtCCqrt, s/CCCrsCCqrtCCqrCCqst * C186 - C11p/Csr,q/Cqs,r/t - C16p/CCrsCCqrt,q/Cqr,r/t,s/Cqs - 187

*187 CCCsrCCqstCCCrsCCqrtCCqrCCqst

§2. Let us suppose that there are some theses which satisfy the given matrix and which are independent of the adopted axiomatic. Among them there must be one thesis which is shortest. I shall show that there is not such a "shortest independent thesis" and, therefore, any thesis which satisfies the given matrix is a consequence of the adopted axiomatic. The proof is as follows:

2.1 As abbreviations I introduce the following symbols:[4]

a) The letters: a, b, c, d and a_1, a_2, a_3, ... a_n ... will denote any senseful expression.

b) The letters: f_1, f_2, f_3, ... f_n ... will denote either the propositional variables or their negation, i.e., "f_1" can denote "p" or "Np" or "Nq", etc.

c) The letters: F_1, F_2, F_3, ... F_n ... will denote either some "f" or the expressions formed by the sign "C" followed by two "f", i.e., "F" can denote "Cf_1f_3" or "Cf_2f_1", etc., where the expression "Cf_1f_3" can denote, e.g., "CpNs" or "CNqNq", etc.

d) The symbol "$a_{(p)}$" will denote, that in "a" there appears one or more times a variable equiform with "p"; consequently "$a_{(-p)}$" will

denote, that in "a" there does not appear such variable; similarly: "$b_{(p)}$", "$F_{n(p)}$", "$f_{(p)}$" etc. Similarly: "$a_{(p,q)}$", "$a_{(-p,q)}$" (in "a" does not appear "p", but appears "q"), etc. Similarly: "$C_{(p)}ab$" will denote, that in "Cab" such variable appears either in "a" or in "b" or in both, etc.

e) It will be said that certain theses belong to the same type, if after replacing their senseful parts by the letters a, b, c, d ... it can be shown that they have some similar formal construction. E.g., the theses 1, 2, 3 belong to the type "CaCbc", but the thesis 45 does not belong.

f) The expression "(a $\rightleftarrows$ b)*(A;B)" will denote, that using only the theses A and B which were proved from the adopted axiomatic in the §1 we can always show that any thesis which belongs to the type a is inferentially equivalent with a thesis which belongs to the type b. Similarly, "(a $\rightleftarrows$ b;c)*(A;B)", "(a $\rightleftarrows$ b;c)*(A;B;C)", etc.

2.2 Below, we shall often use the following two lemmas:

LEMMA I. In the field of the investigated system any thesis of the following type:

$Ca_1Ca_2Ca_3 \ldots Ca_{n-1}Ca_nb$

is inferentially equivalent with a thesis of the following type:

$Ca_nCa_2Ca_3 \ldots Ca_{n-1}Ca_1b$

The proof is apparent from the theses 11 and 12.

LEMMA II. In the field of the investigated system having two theses of the following types:

$Ca_1Ca_2 \ldots Ca_nb$

and

Cbc

we can always obtain a thesis of the following type:

$Ca_1Ca_2 \ldots Ca_nc$

The proof is evident from the theses 1, 11 and 12.

2.3 No expression which consists of the small Latin letters, e.g., p, q, etc. or small Latin letters preceded by optional numbers of negations, e.g., Np, NNq, etc. satisfies the given matrix and, therefore, such expressions cannot be theses in this system.

2.3.1 Thus, any thesis in this system must belong to one of the three following fundamental types:

TI NNa TII NCab TIII Cab

But, any thesis of TI can be proved from a shorter thesis, as

(NNa ⇄ a)*(40;41), and any thesis of TII can be proved from two theses of TIII, as (NCab ⇄ CCabNb; CCabb)*(94;95;96).

Therefore any thesis of TI or TII can be obtained from the theses of TIII.

2.3.2 The theses of TIII belong to one of nine following types:

T1.1	CNNab	**T1.2**	CNCabc	**T1.3**	CCNNabc
T1.4	CCaNNbc	**T1.5**	CCNCabcd	**T1.6**	CCaNCbcd
T1.7	CCCabcd	**T1.8**	CCaCbcd	**T1.9**	CF_1a

Any thesis of T1.1 can be proved from a shorter thesis of T.1, as (CNNab ⇄ Cab)*(60;61).

Any thesis of T1.2 can be proved from a thesis of T1.7, as (CNC abc ⇄ CCCabbCCCabNbc)*(18;87;29;94;95). Therefore, any thesis of T1.2 can be obtained from the theses of T1.3 - T1.9.

Any thesis of T1.3 can be proved from a shorter thesis, as CCNN abc ⇄ CCabc)*(8;40;41) and any thesis of T1.4 can be proved from a shorter thesis, as (CCaNNbc ⇄ CCabc)*(26;40;41). Therefore, any thesis of T1.3 or T1.4 can be obtained from the theses of T1.5 - T1.9.

Any thesis of T1.5 can be proved from a thesis of T1.8, as (CCNC abc ⇄ CCNcCabd)*(1;59) and any thesis of T1.6 can be proved from a thesis of T1.7, as (CCaNCbcd ⇄ CCCbcNad)*(1;58). Therefore, any thesis of T1.5 or T1.6 can be obtained from the theses of T1.7 - T1.9.

Any thesis of T1.7 can be proved from three theses, as (CCCabcd ⇄ CCNacCCbcd;CaCCbcd;CCNbCCNacd)*(102;25;103;159), each of them belongs to TIII and instead of an antecedent of the form "CCabc" possesses two antecedents which are shorter, than the previous one. Therefore, any thesis of T1.7 can be obtained from the theses of T1.8 and T1.9.

Any thesis of T1.8 can be proved from three theses, as (CCaCbcd ⇄ CCaNbCCbcd;CCbNaCCacd;CCacCCbcd)*(115;116;53;170), each of them belongs to TIII and instead of an antecedent of the form "CaCbc" possesses two antecedent which are shorter, than the previous one. Therefore, any thesis of T1.8 can be obtained from the theses of T1.9.

It is evident, that using the theses 1;8;18;25;26;29;40;41;53;58;59;60; 61;94;95;97;102;103;115;116;159;170 suitably we can always obtain any thesis which satisfy the given matrix from some theses of T1.9.

2.3.3 The theses of T1.9 belong to one of the four following types:

T2.1 CF_1NNa T2.2 CF_1NCab T2.3 CF_1f_1 T2.4 CF_1Cab

But, any thesis of T2.1 can be proved from a shorter thesis of T2.2 - T2.4, as (CaNNb ⇄ Cab)*(62;63), and any thesis of T2.2 can be proved from two theses of T2.4, as (CaNCbc ⇄ CaCCbcNc;CaCCbcc)*(1;94; 95;29;97).

It is evident, that in any expression of T2.3 must there appear some variable. Let us assume, that such variable is equiform with "p". Then, it is evident that no expression of T2.3 satisfies the given matrix except the four following:

Cpp; CNpNp; **CCNppp**; **CCpNpNp.**

But these theses are the consequences of our axiomatic, as the theses: 45;72;73. Therefore, any thesis of T1.9 either is a consequence of the adopted axiomatic or can be proved from the theses of T2.4.

2.3.4 Now it is evident, that having any thesis of T2.4, i.e.:

CF_1Cab,

where a possesses a form either NNa or NCac or Cac or F_2 and b possesses a form either NNc or NCcd or Ccd or f_1, then due to the Lemma I we can obtain such thesis from a thesis, which possesses the following form:

$CaCF_1b$,

Using the theses and methods adopted above and the Lemmas I and II we can always obtain such thesis from some theses of the following form:

CF_1CF_2Cab,

where a will be shorter, then a from the previous thesis and b will possess a form either Ccd or f_1. Using such methods and due to the Lemmas I and II we can obtain finally any thesis of T2.4 from the theses of the following type:

T3 $CF_1CF_2CF_3 \ldots CF_nf_1$,

i.e. from such theses in which any antecedent is some F and the last consequence is either some variable or its negation.

Thus, any thesis which satisfies the given matrix either can be obtained from our axiomatic or is a consequence of some theses of T3.

2.3.5 The last sign which appears in the theses of T3 is a variable. Let us assume that such variable is equiform with "q". Then, the theses of T3 belong to one of two following types:

T3.1 $CF_1CF_2 \ldots CF_nNq$

T3.2 $CF_1CF_2 \ldots CF_nq$

But, any thesis of T3.1 we can obtain from some thesis of T3.2 in the following way:

If the variable "q" appears in some F, then these F can have only the following forms:

a) q; b) Nq; c) Cqf_1;
d) $CNqf_1$; e) Cf_1q; f) Cf_1Nq.

Now, if we have a thesis of T3.1 then by substitution for q/Nq we obtain a thesis in which the last consequence possesses the form: NNq and the forms a - f possess - the following:

a_1) Nq; b_1) NNq; c_1) $CNqf_1$;
d_1) $CNNqf_1$; e_1) Cf_1Nq; f_1) Cf_1NNq.

It is evident, that using the Lemmas I and II and the theses 1;40;41; 60;61;62;63 we can everywhere suppress a double negation and obtain some thesis of T3.2. Conversely, having such thesis of T3.2 we can obtain the previous thesis of T3.1.

Therefore, we must only analyse the theses of T3.2.

2.4 However, before further investigations we must prove two theorems which characterize this system and which will be used subsequently:

THEOREM I. No thesis in which some variable appears only once belongs to this system.[5]

Let us suppose that there is such thesis A in which some variable appears only once and which satisfies our matrix. Let us assume that such variable is equiform with "p". Therefore, it is evident that such thesis A belongs to one of the fourteen following types:

Q1	$NNa_{(p)}$	Q2	$NC_{(p)}ab$	Q3	$Cpb_{(-p)}$
Q4	$CNpb_{(-p)}$	Q5	$CNNa_{(p)}b_{(-p)}$	Q6	$CNC_{(p)}abc_{(-p)}$
Q7	$CCa_{(p)}b_{(-p)}c_{(-p)}$	Q8	$CCa_{(-p)}b_{(p)}c_{(-p)}$	Q9	$Ca_{(-p)}NNb_{(p)}$
Q10	$Ca_{(-p)}NC_{(p)}bc$	Q11	$Ca_{(-p)}Np$	Q12	$Ca_{(-p)}p$
Q13	$Ca_{(-p)}Cb_{(p)}c_{(-p)}$	Q14	$Ca_{(-p)}Cb_{(-p)}c_{(p)}$		

But, using some theses which were obtained in §1 we can deduce from any thesis of the types Q1 - Q14 some thesis of Q3. Namely:

From a thesis of Q1 we obtain some thesis of Q2 - Q14, as (NNa → a) *(40). From a thesis of Q2 we obtain some thesis of Q9 - Q14, as (NCab → Cba)*(93). From a thesis of Q4 or Q5 we obtain some thesis of Q3 or Q6 - Q8, as (CNNab → Cab)*(60). From a thesis of Q6 we obtain some thesis of Q9 - 14, as (CNCabc → CNcCab)*(59). From a thesis of Q7 we obtain some thesis of Q3 - Q8 but in which the antecedent is shorter than the antecedent of the previous thesis, as (CCabc → CNaCbc)*(105). From a thesis of Q8 we obtain some thesis of Q3 - Q8, but in which the antecedent is shorter than the antecedent of the previous thesis, as (CCabc → CbCNac)*(106).

Therefore, it is evident that using the above given methods we shall obtain finally from a thesis of Q4 - Q8 some thesis either of Q3 or Q9 - Q14. But, from a thesis of Q9 we obtain a thesis of Q10 - Q14, as (CaNNb → Cab)*(62). From a thesis of Q10 we obtain a thesis of Q7 or Q8, as (CaNb → CbNa)*(58). From a thesis of Q11 or Q12 we obtain some thesis of Q3, as (CaNb → CbNa)*(58). From a thesis of Q13 we obtain some thesis Q3 - Q8, as (CaCbc → CbCac)*(11). From a thesis of Q14 we obtain some thesis of Q4 - Q6, as (CaCbc → CNcCbNa)*(68).

In both these cases the antecedent of the obtained thesis is shorter, than the consequence of the previous thesis. Therefore, it is evident that using the above given methods we shall obtain finally from a thesis of Q9 - Q14 some thesis of Q3 - Q8. Therefore, using the theses 11; 40;58;59;60;62;68;93;105;106 we can always deduce from a thesis of Q1 - Q14 some thesis of Q3.

Thus, if we suppose that in our system there is the thesis A, then we must adopt that some thesis B of Q3 belongs to this system. But any thesis of Q3 does not satisfy the given matrix which we can easily prove

by substituting in such theses for p/1 and for any variable which is not equiform with "p" - 2. Then, we get: C12 = 0.

Therefore, the thesis B does not belong to our system and therefore the thesis A cannot belong to our system and our supposition was false. Then, we have proved our theorem.

THEOREM II. Any thesis in which there appears only one variable is a thesis of our system.

Let us assume that some thesis A possesses such property and its sole variable is equiform with "p". Therefore, either this thesis A is a consequence of the adopted axiomatic, as, e.g., the theses 40;41;45; 72;73, or it can be obtained by the methods which have shown above from some set K of the theses of T3.2. But, it is evident that in any thesis which belongs to this set K there appear only such variables which are equiform with "p". Therefore, any of such theses possesses the following form:

$$CF_1CF_2 \ldots CF_np,$$

where the expressions F_1, $F_2 \ldots F_n$ belong to one of the following forms:

B1 p; B2 Np; B3 Cpp; B4 CNpNp; B5 CNpp; B6 CpNp.

But, any thesis of T3.2 in which there appear some expressions of B6 can be proved from some thesis of T3.2 without such expressions, as $(CF_1CF_2 \ldots CF_kCCpNpCF_{k+2} \ldots CF_nq \rightleftarrows CNpCF_1CF_2 \ldots CF_kCF_{k+2} \ldots CF_nq)$*(Lemma I;1;43;73). Similarly, any thesis of T3.2 in which there appear some expressions of B5 can be proved from some thesis of T3.2 without such expressions, as $(CF_1CF_2 \ldots CF_kCCNppCF_{k+2} \ldots CF_nq \rightleftarrows CpCF_1CF_2 \ldots CF_kCF_{k+2} \ldots CF_nq)$*(Lemma I;1;34;72). Also, any thesis of T3.2 in which there appear some expressions of B4 can be proved from two theses T3.2 without such expressions, as $(CF_1CF_2 \ldots CF_kCCNpNpCF_{k+2} \ldots CF_nq \rightleftarrows CpCF_1CF_2 \ldots CF_kCF_{k+2} \ldots CF_nq; CNpCF_1CF_2 \ldots CF_kCF_{k+2} \ldots CF_nq)$*(Lemma I;1;35;40;41; 44;76). And similarly, any thesis of T3.2 in which there appear some expressions of B3 can be proved from two theses of T3.2 without such expressions, as $(CF_1CF_2 \ldots CF_kCCppCF_{k+2} \ldots CF_nq \rightleftarrows CNpCF_1CF_2 \ldots CF_kCF_{k+2} \ldots CF_nq; CpCF_1CF_2 \ldots CF_kCF_{k+2} \ldots CF_nq)$*(Lemma I;1;35;44;76).

Therefore, any thesis of T3.2 can be proved from some theses of T3.2 which do not possess the expressions of the forms B3,B4,B5,B6 (evidently, for any kind of the variables) and any thesis from the set K can be obtained from some theses of T3.2 in which there appear only the variables equiform with "p" and only the expressions of the forms B1 or B2. But, any such thesis is a consequence of the adopted axiomatic, because:

a) if it possesses two equiform antecedents, then they can be reduced to one, as $(CaCab \rightleftarrows Cab)$*(Lemma I;3;46);

b) if it does not possess such antecedents, then it can have only the following forms:

Cpp; CNpCpp; CpCNpp.

But, these forms are the theses of our system: 45;35;34. Therefore, any thesis from the set K can be proved from the adopted axiomatic and, consequently, also the thesis A.

Thus, because the thesis A can be any thesis which satisfies the condition of our theorem, we have proved this theorem.

2.5 Now, we can return to our analysis of the theses of T3.2. Using entirely similar methods to those which were adopted for the demonstration of the Theorem II, we can easily prove that any thesis of T3.2 can be obtained from some theses of T3.2, but in which:

a) there are no expressions of the forms B3 - B6, not only for the variable p, but also for any other variables;

b) There are not two equiform antecedents.

This is due to the Lemma I and the theses 1;3;34;35;40;41;43;44;45;46; 72;73;76.

2.5.1 Let us assume the theses of T3.2 which are the properties a and b, as the theses of the type:

T4 $CF_1CF_2 \ldots CF_nq$,

where the expressions F_1, F_2 ... etc. can possess only the following forms:

D1 p; D2 Np; D3 CNpr; D4 CpNr

(here the variables p and r are given for example only; in these forms, evidently, any variable can appear).

Thus, any thesis which satisfies our matrix either is a consequence of the adopted axiomatic or can be obtained from some theses of T4. We know that due to our Lemma I we can order in an optional way the antecedents of the theses of T4 and we know also due to the Theorem I that if the variable q is a last sign of such theses, then it must also appear at least in one antecedent of each of them. Therefore, the theses of T4 belong to one of the eight following types:

T4.1 $CrCNrC(-r)F_1CF_2 \ldots CF_nq$
T4.2 $CF_1CF_2 \ldots CF_nCNqq$
T4.3 $CF_1CF_2 \ldots CF_nCqCCqf_1q$
T4.4 $CF_1CF_2 \ldots CF_nCqCCNqf_1q$
T4.5 $CF_1CF_2 \ldots CF_nCqCCf_1qq$
T4.6 $CF_1CF_2 \ldots CF_nCqCCf_1Nqq$
T4.7 $CF_1(-q)CF_2(-q) \ldots CF_n(-q)Cqq$
T4.8 $CF_1CF_2 \ldots CF_{n-1}CF_nq$

(in the expressions of T4.8 no F in which there appears a variable equiform with "q" possesses a form D1 or D2).

No expression of T4.1 satisfies our matrix, if its consequence does not satisfy this matrix: C2CN20 = C2C20 = C20 = 0. Therefore, if there is some expression of T4.1 which is a thesis of our system, e.g.:

A) $CrCNrC(-r)F_1CF_2 \ldots CF_nq$,

then also the following shorter expression:

B) $C(-r)F_1CF_2 \ldots CF_nq$,

also of T4 belongs to this system, But, using the thesis 4 we can obtain A from B.

Any thesis of T4.2 can be obtained from a shorter thesis of T4.3 - T4.8, as $(CNaa \rightleftarrows a)^*$(Lemma II;34;72). Any thesis of T4.3 can be obtained from a shorter thesis of T4.3 - T4.7, as $(CaCCaba \rightleftarrows CbCaa)^*$ (Lemma II;99;108). Any thesis of T4.4 can be obtained from a shorter thesis of T4.3 - T4.7, as $(CaCCNaba \rightleftarrows CbCaa)^*$(Lemma II;101;111). Any thesis of T4.5 can be reduced to same thesis of T4.4 as $(Cab \rightleftarrows CNbNa)^*$(Lemma I;1;5;40;41;64). Any thesis of T4.6 can be reduced to same thesis of T4.3, as $(CaNb \rightleftarrows Cba)^*$(Lemma I;1;40;41;58). Therefore, it is evident that any thesis of T4.3 - T4.6 we can obtain from some thesis of T4.7.

But, any thesis of T4.7 is inferentially equivalent to some thesis of the following form:

A) $CqCNqC(-q)F_2CF_3 \ldots CF_nNF_1$ (Lemmas I and II;1;5;64)

and if an expression of the form A satisfies our matrix, then also its consequence of the following form:

B) $C(-q)F_2CF_3 \ldots CF_nNF_1$

must be a thesis of our system. Contrarily, we have: $C2CN20 = C2C20 = C20 = 0$. But, using the thesis 4 we can obtain A from B and from the Lemmas I and II and the theses 1;40;41;60;61;62;63;94;95;97 we know that any thesis of the form B is inferentially equivalent to one or two theses of T3.2 which are shorter than the previous thesis of the form A. But, any thesis of T3.2 either is a consequence of our axiomatic or can be obtained from some theses of T4.

Thus, as a final result we have proved that any thesis of T4 either can be obtained from some shorter theses of T4 or it is a thesis of T4.8. Therefore, any thesis of T4 can be reduced to some theses of T4.8.

2.5.2 Any thesis of T4.8 possesses the property that if its last sign is, e.g., equiform with "q", then an expression F which belongs to this thesis and in which there appears a variable equiform with "q" belongs to one of the four following forms:

E1 Cqf_1; E2 $CNqf_1$; E3 Cf_1q; E4 Cf_1Nq.

But, using the Lemma I and the theses 1;5;58;62;63;64 we can always obtain some thesis of T4.8 which will be inferentially equivalent with the previous thesis and in which there will be no expression F of the forms E3 or E4. Let us assume the type of such theses, as T5.

Any expression of T5 belongs to one of the five following types:

T5.1 $CF_{1(-q)}CF_{2(-q)} \ldots CF_{n(-q)}CCqf_1CCqf_2 \ldots CCqf_kq$
T5.2 $CF_1CF_2 \ldots CF_nCCNqf_1CCNqf_2 \ldots CCNqf_kq$

(in the expressions of T5.2 no F in which there appears a variable equiform with "q" possesses a form E2)

T5.3 $CF_1(-q)CF_2(-q) \ldots CF_n(-q)CCNqf_1CCqf_2CCqf_3 \ldots CCqf_kq$
T5.4 $CF_1(-q)CF_2(-q) \ldots CF_n(-q)CCNqf_1q$
T5.5 $CF_1(-q)CF_2(-q) \ldots CF_n(-q)CCNqf_1CCqf_2q$

No thesis from our system belongs to the type T5.1, because no expression of T.51 satisfies our matrix. It is evident, if in such expression we substitute for q/0 and for any other variable - 2. Then we have: C2C2 ... C2CC02CC02 ... CC020 = C2C2 ... C2C1C1 ... C10 = C2 C2 ... C20 = C20 = 0.

Any thesis of T5.2 can be obtained from some theses of T5.3, T5.4 or T5.5, as (CCabCCacd $\rightleftarrows$ CCcbCCacd;CCbcCCabd)*(Lemmas I and II; 1;29;30;31;187), and using the same theses and methods we can obtain any thesis of T5.3 from some theses of T5.4 or T5.5.

Thus, we have proved that any thesis which satisfies our matrix either is a consequence of the adopted axiomatic or belongs to the types T5.4 or T5.5.

2.5.3 Any thesis of T5.4 is inferentially equivalent with some thesis of the following type:

T6.1 $CNqCCNqf_1C(-q)F_2CF_3 \ldots CF_nNF_1$ (Lemmas I and II; 5;64)

And any thesis of T5.5 is inferentially equivalent with some thesis of the following type:

T6.2 $CNqCCNqf_1CCqf_2C(-q)F_2CF_3 \ldots CF_nNF_1$ (Lemmas I and II;5;64)

But, from any thesis of T6.1 we can deduce a thesis of

T6.11 $CNqCqC(-q)f_1CF_2CF_3 \ldots CF_nNF_1$,

as (CCNabc $\rightleftarrows$ CaCbc)*(Lemma I;3;104;105). But, if some expression of T.6.11 is a thesis, then also a shorter expression being its consequence of the following type:

T6.111 $C(-q)f_1CF_2CF_3 \ldots CF_nNF_1$

must satisfy our matrix, i.e., it must be a thesis of this system. Because, if for some substitutions such consequence = 0, then also a whole thesis = 0, if we substitute for q/2, as: CN2C20 = C2C20 = C20 = 0. But, if such consequence of T6.111 is a thesis, then from it we can always obtain a previous thesis of T.6.1, as (Cab $\rightarrow$ CcCCcab)*(24). Therefore, any thesis of T.6.1 we can imply from some shorter thesis of T.6.111 in which no variable is equiform with "q", and which we can easily transform into some not longer thesis of T.3 (Lemmas I and II; 1;5;40;41;64). But, as we know, any thesis of T.3 either is a consequence of the adopted axiomatic or is inferentially equivalent with some not longer theses of T.6.1 or T.6.2.

Thus, it is evident that using the theses and methods shown above

we can always prove that any thesis which satisfies our matrix either is a consequence of the adopted axiomatic or is implied from some theses of T.6.2.

2.5.4 Using the Lemmas I and II, the theses 1;40;41;60;61;62;63;94; 95;97 and the methods which were applied above several times we can always prove that any thesis of T.6.2 is inferentially equivalent with one or two theses of the following type:

T7 $CNqCCNqf_1CCqrC_{(-q)}F_1CF_2 \dots CF_nf_m$

(where the expression f_m is some variable or its negation and instead of the expression f_2 in the theses of T6.2 there is some optional variable, e.g., "r" which is unequiform with "q").

For shortening let the symbols: $M_1, M_2, \dots M_n$, etc. for any expression of T3. E.g.: $M_1 = CF_1CF_2 \dots CF_nf_m$.

From Theorem I we can establish that any expression of T7 belongs to one of the five following types:

T7.1 $CNqCCNqrCCqrM_{1(-q,-r)}$
T7.2 $CNqCCNqNrCCqrM_{1(-q,-r)}$
T7.3 $CNqCCNqrCCqrM_{1(-q,r)}$
T7.4 $CNqCCNqNrCCqrM_{1(-q,r)}$
T7.5 $CNqCCNqf_{1(-r)}CCqrM_{1(-q,r)}$

No expression of T7.1 satisfies our matrix, since substituting for r/1 and for any other variable - 2 we get: $CN2CCN21CC212 = C2C1C12 = C2C10 = C20 = 0$ and no expression of T7.2 satisfies our matrix, since substituting for r/0, for q/0 and for any other variable - 2 we get: $CN0CCN0N0CC002 = C1CC11C12 = C1C10 = C10 = 0$. Therefore, no thesis belongs to T7.1 or T7.2.

If some thesis of T7.3 satisfies our matrix, then also its consequence of the following type:

T7.31 $CNqCqCrM_{1(-q,r)}$ (Lemma I;3;104;105)

satisfies our matrix. But, if some expression of T7.31 is a thesis of our system, then the following expression must also be a thesis of this system:

T7.32 $CrM_{1(-q,r)}$,

since: $CN2C20 = C20 = 0$. However, from a suitable thesis of T7.32 we can always obtain the previous thesis of T7.3 in the following way:

A) $CrM_{1(-q,r)}$ (T7.32)
B) $CNqCCNqrM_{1(-q,r)}$ (24;A)
C) $CNqCCNqrCNqM_{1(-q,r)}$ (46;B;Lemma I)
D) $CNqCCNqrCrM_{1(-q,r)}$ (49;B;Lemma I)
E) $CNqCCNqrCCqrM_{1(-q,r)}$ (76;C;D;Lemma I)

Therefore, any thesis of T7.3 can be obtained from some shorter thesis of T3.

If some expression of T7.4 satisfies our matrix, then also a suitable expression:

T7.41 $CNrM_{1(-q,r)}$

satisfies our matrix.

Let us suppose that it is false. Then, there is some expression of T7.4, e.g.:

A) $CNqCCNqNrCCqrM_{2(-q,r)}$,

which satisfies, and a suitable expression of T7.41:

B) $CNrM_{2(-q,r)}$,

which does not satisfy our matrix. If the expression B is false, then for some substitutions of 0, 1 or 2 for all its variables this expression = 0. Since the variable "r" appears in $M_{2(-q,r)}$ any of such substitutions can give one of the two following forms:

B_1) CN00 $\qquad$ B_2) CN20

But, any of these same substitutions in the expression A convert s it so that it becomes false, as:

A_1) CNqCCNqN0CCq00 = 0, for q/0 $\quad$ A_2) CNqCCNqN2CCq20 = 0, for q/2

Thus, our assumption leads to the contradiction and, therefore, our lemma is true. Then, the expression B is also true. But, from the expression B we can always obtain the expression A in the following way:

C)	$CNqCCNqNrM_{2(-q,r)}$	(24;B)
D)	$CNqCCNqNrCNqM_{2(-q,r)}$	(46;C;Lemma I)
E)	$CNrCrM_{2(-q,r)}$	(81;B;Lemma I)
F)	$CNqCCNqNrCrM_{2(-q,r)}$	(24;E)
A)	$CNqCCNqNrCCqrM_{2(-q,r)}$	(76;D;F;Lemma I)

Therefore, any thesis of T7.4 can be obtained from some shorter thesis of T3 and any thesis which satisfies our matrix either is a consequence of the adopted axiomatic or belongs to the type T7.5.

2.5.5 Any thesis of T7.5 is inferentially equivalent with some thesis of the following type:

T8 $CNqCCNqsCCqrM_{1(-q,r,s)}$

(Theorem I;Lemmas I and II;1;5;40;41;58;59;64)

(in the expressions of T8 the form M possesses the following structure:

$CF_1CF_2 \ldots CF_nf_m$

and if "r" appears in some F, then this F possesses one of the follow-

ing forms: r;Nr;Crf_k;$CNrf_k$; and if "r" appears in f_m, then this f_m is either r or Nr).

Using the Lemmas I and II and the theses 1;40;41;94;95;97 we can always prove that any expression of T8 is inferentially equivalent with some expressions of the following types:

T8.1 $CNqCCNqsCCqrC_{(-q,s)}CNrf_1CCNrf_2 \ldots CCNrf_nM_{1(-r)}$
T8.2 $CNqCCNqsCCqrCrM_{1(-q,s)}$
T8.3 $CNqCCNqsCCqrCNrM_{1(-q,s)}$
T8.4 $CNqCCNqsCCqrC_{(-q,s)}CNrf_1CCNrf_2 \ldots CCNrf_nM_{1(r)}$

(where in M the variable "r" appears only in the expressions of the form "Crf_k")

T8.5 $CNqCCNqsCCqrC_{(-q,s)}CNrf_1CCrf_2CCrf_3 \ldots CCrf_nM_{1(-r)}$
T8.6 $CNqCCNqsCCqrC_{(-q,s)}Crf_1CCrf_2 \ldots CCrf_nM_{1(-r)}$
T8.7 $CNqCCNqsCCqrC_{(-q,s)}CNrf_1CCrf_2M_{1(-r)}$
T8.8 $CNqCCNqsCCqrC_{(-q,s)}Crf_1M_{1(-r)}$

No expression of T8.1 is a thesis of our system, since substituting in such expression for any variable - which is unequiform with "r" - 2 and for r/1 we get: CN2CCN22CC21CCN12CCN12 ... CCN122 = C2C2C1C1 ... C12 = C2C20 = 0.

If some expression of T8.2 satisfies our matrix, then also its consequence of the following type:

T8.21 $CNqCqCsCrM_{1(-q,s)}$ (104;105;3;Lemma I)

satisfies our matrix. But, if some expression of T8.21 is a thesis of our system, then a suitable expression of the following type:

T8.22 $CsCrM_{1(-q,s)}$

is also a thesis, since, contrarily: CN2C20 = C20 = 0. However, from such thesis of T8.22 we can always obtain the previous thesis of T8.2, as:

A) $CsCrM_{1(-q,s)}$ (T8.22)
B) $CNqCCNqsCrM_{1(-q,s)}$ (24;A)
C) $CNqCCNqsCNqCrM_{1(-q,s)}$ (46;B;Lemma I)
D) $CsCrCrM_{1(-q,s)}$ (46;A;Lemma I)
E) $CNqCCNqsCrCrM_{1(-q,s)}$ (24;D)
F) $CNqCCNqsCNqrCrM_{1(-q,s)}$ (76;C;E;Lemma I)

Therefore, any thesis of T8.2 can be obtained from some shorter thesis of T8.22, i.e., from some shorter thesis of T3.

If some expression of T8.3 satisfies our matrix then also a suitable expression of the following type:

T8.31 $CCNsrCNrM_{1(-q)}$

satisfies our matrix.

Let us suppose that it is false. Then there is some expression of T8.3,

e.g.:

A) $CNqCCNqsCCqrCNrM_{2(-q)}$

which satisfies and there is a suitable expression of T8.31:

B) $CCNsrCNrM_{2(-q)}$

which does not satisfy our matrix. Therefore, for some substitutions of 0, 1 or 2 for all its variables this expression $B = 0$. Since the variable "s" must appear in $M_{2(-q)}$ (Theorem I) any of such substitutions can give one of the three following forms:

B_1) CCN10CN00 B_2) CCN12CN20 B_3) CCN22CN20

But, any of these same substitutions in the expression A converts it so that it becomes false, as:

A_1) $CNqCCNq1CCq0CN00 = 0$, for q/0
B_2) $CNqCCNq1CCq2CN20 = 0$, for q/2
C_3) $CNqCCNq2CCq2CN20 = 0$, for q/2

Thus, our assumption leads to contradiction and, therefore, our lemma is true. But, from the expression B we can obtain the expression A, as:

C) $CCNsqCCqrCNrM_{2(-q)}$ (8;11;B)
D) $CCNqsCCqrCNrM_{2(-q)}$ (1;59;C)
A) $CNqCCNqsCCqrCNrM_{2(-q)}$ (47;D)

Therefore any thesis of T8.3 can be obtained from some shorter thesis of T8.31, i.e., from some shorter thesis of T.3.

Using the theses 30;31;187 and the Lemma I we can reduce any thesis of T8.4, T8.5 or T8.6 to the some not longer theses of T8.7 or T8.8. Thus, any thesis which satisfies our matrix either is a consequence of the adopted axiomatic or belongs to the type T8.7 or T8.8.

2.5.6 Using the Lemma I and the theses 1;5;30;31;64;187 we can reduce any thesis of T8.7 or T8.8 to one or two not longer theses of the following type:

T9 $CCf_{1(-r)}rCCrf_{2(-r)}M_{1(-r)}$

(where the variables which appear in f_1 and f_2 are unequiform with "r" but can be equiform with "q" and they must appear in $M_{1(-r)}$; conversely such an expression of T9 does not satisfy our matrix, as: $CC00CC002 = 0$ or $CC00CC012 = 0$ (Theorem I)).

If some expression of T9 satisfies our matrix, then also a suitable expression of the following type:

T9.1 $CCf_{1(-r)}f_{2(-r)}M_{1(-r)}$

satisfies our matrix.

Let us suppose that it is false. Then there is some expression of T9, e.g.:

A) $CCf_{1(-r)}rCCrf_{2(-r)}M_{2(-r)}$

which satisfies and there is a suitable expression of T9.1:

B) $CCf_{1(-r)}f_{2(-r)}M_{2(-r)}$

which does not satisfy our matrix. Therefore for some substitutions of 0, 1 or 2 for all its variables this expression $B = 0$. Since the variables which appear in f_1 and f_2 must also appear in $M_{2(-r)}$, any of such substitution can be one of the following forms:

B_1) CC000 B_2) CC010 B_3) CC020
B_4) CC110 B_5) CC210 B_6) CC220

But, any of these same substitutions in the expression A converts it so that it becomes false, as:

A_1) $CC0rCCr00 = 0$, for r/0
A_2) $CC0rCCr10 = 0$, for r/0
A_3) $CC0rCCr20 = 0$, for r/0
A_4) $CC1rCCr10 = 0$, for r/1
A_5) $CC2rCCr10 = 0$, for r/1
A_6) $CC2rCCr20 = 0$, for r/2

Thus, our assumption leads to contradiction and, therefore, our lemma is true. But, from the expression B we can obtain the expression A, as:

A) $CCf_{1(-r)}rCCrf_{2(-r)}M_{2(-r)}$ (8;11;B)

Therefore, any thesis of T9 can be obtained from some shorter thesis of T9.1 in which there appears a variable equiform with "q" but does not appear a variable equiform with "r". Any thesis of T9.1 either belongs to T3 or can be easily by above given methods transformed to some not longer thesis of T3. But, any thesis of T3 either is a consequence of our axiomatic or can be deduced from some not longer theses of T8 and, consequently, from some not longer theses of T9. Because any thesis of T9 can be deduced from some shorter thesis of T3, then gradually we can deduce such theses from shorter and shorter theses of this type. Then, there are no theses which satisfy our matrix and are not a consequence of the adopted axiomatic. Therefore, there is not a "shortest independent thesis" and, therefore, this axiomatic is adequate for our matrix.

2.6 On the other hand, let us assume that an expression:

A) $R_1(p,q,r, \ldots.)$

symbolizes any senseful expression constructed from the functors "C" and "N" and variables: p, q, r, etc. but which is not a thesis of this

system, i.e., for some substitutions of 0, 1 and 2 for its variables this expression A does not satisfy the given matrix.

If we have the expression A, then we also have the following expression:

B) $R_1(p,Cqq,NCqq)$

which is obtained from the expression A in the following way:

Knowing that some substitutions of 0 for some variables and some substitutions of 1 and 2 for another variables show that the expression A is no thesis of this system, we substitute for every variable for which we before substituted 2 a variable "p", for every variable for which we before substitute 1 - an expression "Cqq", for any other variable - an expression "NCqq". It is evident that the expression B also does not satisfy our matrix, if we shall subsitute for q/0 and for p/2.

But, an expression:

C) $CR_1(p,Cqq,NCqq)CpCNpq$

is a thesis of our system, because for any substitution of 0, 1 and 2 for its variables "p" and "q" this expression C satisfy the given matrix. Therefore, as it has established above the thesis C can be deduced from the adopted axiomatic.

But, from the expressions B and C we obtained at once an expression:

D) CpCNpq

which does not satisfy the given matrix but which is a thesis of the bi-value calculus of propositions. Since in the field of this system we have the theses 1 and 72, then having also the thesis D we get a complete system of the calculus of propositions. Therefore, by adding to our system any senseful expression we obtain at least the bi-value system. It shows that the investigated system possesses the 3-rd degree of extension.[6]

Thus, it has been shown that any thesis which satisfies the given matrix is a consequence of the adopted axiomatic and the addition of any thesis which does not satisfy this matrix to the adopted axiomatic converts this system into a complete bi-value system of the calculus of propositions.

§3. The following matrices show that each of the adopted axioms is independent from the others:

3.1 The independency of the axiom 1 is proved by a matrix:

	C	0	1	2	3	N
	0	1	1	1	1	1
*	1	0	1	2	2	0
	2	1	1	1	1	3
	3	0	1	1	1	3

in which the designated value is 1. This matrix satisfies the axioms 2, 3, 4 and 5, but not 1, e.g., if we substitute in 1 for p/3, for q/2 and for r/0: CC32CC20C30 = C1C10 = C10 = 0.

3.2 The independency of the axiom 2 is proved by a matrix:

	C	0	1	2	N
	0	1	1	1	1
*	1	0	1	0	0
	2	0	1	1	2

in which the designated value is 1. This matrix satisfies the axioms 1, 3, 4 and 5, but not 2, e.g., if we substitute in 2 for p/2 and for q/2: C2CC222 = C2C12 = C20 = 0.

3.3 The independency of the axiom 3 is proved by a matrix:

	C	0	1	2	N
	0	1	1	0	0
*	1	0	1	2	2
	2	1	1	1	1

in which the designated value is 1. This matrix satisfies the axioms 1, 2, 4 and 5, but not 3, e.g., if we substitute in 3 for p/0 and for q/2: CC0C02C02 = CC000 = C10 = 0.

3.4 The independency of the axiom 4 is proved by a matrix:

	C	0	1	2	N
	0	1	1	1	1
*	1	0	1	0	0
	2	0	1	0	2

in which the designated value is 1. This matrix satisfies the axioms 1, 2, 3 and 5, but not 4, e.g., if we substitute in 4 for p/2 and for q/2: C2C2CN22 = C2C2C22 = C2C20 = C20 = 0.

3.5 The independency of the axiom 5 is proved by a matrix:

	C	0	1	N
	0	1	1	0
*	1	0	1	0

in which the designated value is 1. This matrix satisfies the axioms 1, 2, 3 and 4, but not 5, e.g., if we substitute in 5 for p/0 and for q/1: CCN0N1C10 = CC000 = C10 = 0.

§4. In this paragraph some relations between our system and some other systems of the calculus of propositions will be outlined.

4.1 From §2.6 we know that the adding to the investigated system of any thesis which belongs to bi-value system, but does not satisfy our matrix converts this system into a complete two-value system. The addition of some theses cause this axiomatic to cease to be independent. Thus, e.g., if we shall add a thesis:

I CpCqp,

then we can obtain the theses 2 and 4 from the theses 1, 3 and I. But, there are some theses which do not possess this property, e.g., a thesis:

II CCppCqCpp

The set of the theses 1, 2, 3, 4, 5 and II constitutes a complete axiomatic of the bi-value system. The independency of each of these theses from

the others is shown by the matrix of our system and the matrices given in §3.

4.2 It is well known that the theses 1 and 12 are independent each from the other.[7] We can prove this by the following matrices:

A)

	C	0	1	2	3
	0	1	1	3	1
*	1	0	1	0	0
	2	0	1	0	0
	3	0	1	3	1

B)

	C	0	1	2	3
	0	1	1	1	1
*	1	0	1	2	3
*	2	3	1	2	3
	3	0	1	2	2

The matrix A in which the designated value is 1 satisfies the thesis 1, but not the thesis 12, as: CC32CC03C02 = C3C13 = C30 = 0. The matrix B in which the designated values are 1 and 2 satisfies the thesis 12, but not the thesis 1, as: CC12CC20C10 = C2C30 = C20 = 3.

But, the system of the theses 1 and 2 is inferentially equivalent with a system of theses 12 and 2. The proof that the theses 1 and 2 imply 12 is given in §1. Now, we shall obtain 1 from 12 and 2:

12 CCqrCCpqCpr
2 CpCCpqq
 12p/q,q/CCqrr,r/CCpCqrCpr * 12q/Cqr - C2p/q,q/r - III
III CqCCpCqrCpr
 IIIp/CpCqr,q/CqCCpCqrCpr,r/CqCpr * CIII - CIIIp/q,q/CpCqr,r/Cpr - 11
11 CCpCqrCqCpr
 11p/Cqr,q/Cpq,r/Cpr * C12 - 1
1 CCpqCCqrCpr

Therefore, the set of the theses 12, 2, 3, 4, 5 is also an axiomatic of our system. The matrices from §3 shown the independency of these theses, because the matrices from §3.2-3.5 satisfy the thesis 12, but matrix from §3.1 does not, as: CC20CC32C30 = C1C10 = C10 = 0.

4.3 Being a partial system of the two-value logic this system belongs also to the three-value calculus of propositions, because it cannot possess an adequate two-value matrix. But this system is essentially different from the three-value calculus which belongs to the family of the many-value systems of J. Łukasiewicz. Using the notation adopted

above we can define this system of Łukasiewicz by the following matrix C:[8]

c)

C	0	1	2	N
0	1	1	1	1
* 1	0	1	2	0
2	2	1	1	2

D)

p	Tp
0	2
1	2
2	2

In this matrix the designated value is 1, and this matrix determines a partial system for which an adequate axiomatic has been given by M. Wajsberg:[9]

I CpCqp
1 CCpqCCqrCpr
IV CCCpNppp
5 CCNpNqCqp

As can be easily shown, this system also possesses the 3-rd degree of extension, and there are many theses which are common to both systems, e.g.: 1, 2, 4, 5, IV, etc. But there are some theses which hold only in our system, e.g., 3 (CC2C20C20 = CC222 = C12 = 2) or 72 (CCN222 = CC222 = C12 = 2), and some which hold only in Łukasiewicz's system, e.g., I (Theorem I). Also, there are the theses which do not belong to any of these systems, as, e.g., the law of Peirce:

V CCCpqpp

(Theorem I and CCC2022 = CC222 = C12 = 2). The fact that in Łukasiewicz's system the thesis I holds, but not 72, shows that not both Theorems I and II can be applied to this system. The fact that both these systems possess the same degree of extension shows that it is impossible to interpret each of these systems into the other.

4.4 It is known that:
For any integer n there are n-1 different and complete n-value systems of the calculus of propositions; these systems possess the same degree of extension and cannot be translated one into the other.

Thus, there is only one complete system of two-value calculus, while three-value logic possesses two complete and different systems. One of them has been established by J. Słupecki who has completed Łukasiewicz's system:[10]

1) by adding a new functor, Tp, for one argument which is defined by the matrix D,
and

2) by adding to the Wajsberg's axiomatic the following two new axioms:

VI CTpNTp
VII CNTpTp

I have discovered the second system proving that the following matrix E:

E)

	C	0	1	2	H
	0	1	1	2	2
*	1	0	1	2	0
*	2	0	1	1	1

F)

	C	0	1	2	H
	0	1	1	1	2
*	1	0	1	0	0
*	2	0	1	2	1

in which the designated values are 1 and 2, and its adequate axiomatic:

VIII CCCpqrCCrpCsp
IX CCHqCHHqHpCHHpCpq
X CCHHpHHqHHCpq
XI CHHCpqCHHpHHq

with the usual rules of reasoning constitute the second system.[11]

Both these systems (Słupecki's and mine) possess the 2-nd degree of extension, have different properties and cannot be transformed one into the other. It is obvious that the functors "C" and "N" which appear in the investigated system cannot be defined by any combination of the functors "C", "N" and "T" from Słupecki's system, because it is impossible having only one designated value to define two such values. For a similar reason the functors "C" and "N" from Łukasiewicz's system cannot be defined in my system. But, the functors "C" and "N" from our investigated system can be defined in my complete system, e.g., in the following way:

In order to distinguish both "C"'s, i.e., the "C" from our system and the "C" from the complete system, I put "C_1" as a symbol of the second "C". Then:

D I $Np = C_1C_1HC_1ppHpHHC_1HC_1ppHHp$
D II $Cpq = C_1HC_1C_1HC_1ppHpC_1HC_1qqHHC_1C_1HC_1qqHqHqHC_1HC_1HC_1pppHC_1qHHC_1qq$

On the other hand, if we take the functor "C" and "H", then we can define not only the functor "N", but also "C_1", as:

D III $Np = HHCHHpHHCpp$
D IV $C_1pq = CHHCCCHpHpHHCppCHqHqCCCHHCppHHCppCqqCCHHCppHHCppq$

It shows that the system defined by the matrix F is equivalent with

the system defined by the matrix E, i.e., we can adopt instead of the functors "C_1" and "H" the functors "C" and "H", as primitive terms of my complete system.

4.5 The difference between our system and the three-value system of Łukasiewicz appears much more obvious, if we add to our system Słupecki's functor "T" adjusted to our matrix. In the field of Łukasiewicz's system the expressions:

XII Tp
XIII NTp

are not theses, as: T2 = 2 and NT2 = N2 = 2. In the field of our system, where 2 is a designated value, both these expressions are true and can be added to the adopted axiomatic as new axioms. Then we get a system in which we can easily obtain theses VI and VII, and we can define the functor "N", as:

D V Np = CpTp

but, in the field of such system we cannot define the functor "H" and, therefore such a system is only a partial system of my complete three-value system.

Whereas, by the addition to the Łukasiewicz's system of theses VI and VII (which are weaker, than XII and XIII) this system is converted into the complete system of Słupecki.

4.6 Evidently, the investigated system does not belong to the so called systems of strict implication and its functor "C" cannot be interpreted as some kind of such implication. But there is some resemblance between these systems. Almost all, so called, "paradoxical" theses[12] do not hold in our system, namely:

In our system we can define conjunction, equivalence and alternative by the following definitions:

D VI Kpq = NCpNq
D VII Epq = KCpqCqp
D VIII Apq = CNpq

Then these functors have the matrices:

G)

	K	0	1	2
	0	0	0	0
*	1	0	1	1
*	2	0	1	2

H)

	E	0	1	2
	0	1	0	0
*	1	0	1	0
*	2	0	0	2

I)

	A	0	1	2
	0	0	1	0
*	1	1	1	1
*	2	0	1	2

and the following "paradoxical" theses do not hold in our system:

I	(7.4)	CpCqp	(Theorem I)
XIV	(7.41)	CNpCpq	(Theorem I)
XV	(7.5)	CKpqCpq	(CK12C12 = C10 = 0)
XVI	(7.5)	CKpqCqp	(CK21C12 = C10 = 0)
XVII	(7.51)	CKNpNqCpq	(CKN2N0C20 = CK210 = C10 = 0)
XVIII	(7.51)	CKNpNqCqp	(CKN0N2C20 = CK120 = C10 = 0)
XIX	(7.6)	CNCpqq	(Theorem I)
XX	(7.61)	CNCpqNq	(Theorem I)
XXI	(7.62)	CNCpqCpNq	(CNC12C1N2 = CN0C12 = C10 = 0)
XXII	(7.63)	CNCpqCNpq	(CNC20CN20 = CN0C20 = C10 = 0)
XXIII	(7.7)	CKpqEpq	(CK12E12 = C10 = 0)
XXIV	(7.71)	CKNpNqEpq	(CKN0N2E02 = CK120 = C10 = 0)

Only four such theses belong to our system:

XXV	(7.52)	CKNpqCpq
XXVI	(7.64)	CNCpqCNpNq
93	(7.65)	CNCpqCqp
XXVII	(7.72)	CKpNqNEpq

On the other hand, the following, e.g., "non-paradoxical" theses do not belong to our system:

XXVIII	CKpqp	(CK212 = C12 = 0)
XIX	CKpqq	(CK122 = C12 = 0)
XXX	CpApq	(C2A20 = C20 = 0)
XXXI	CpAqp	(C2A02 = C20 = 0)

4.7 The question remains open whether a system defined by the im - plicational part of our matrix possesses a finite axiomatic and what is its degree of extension. The following matrix:

J) *

C	0	1	2	3
0	1	1	1	1
1	0	1	2	3
2	1	1	1	1
3	0	1	0	1

in which the designated value is 1 shows that a thesis:

XXXII CCpCCqqpCCCpqpp

is not a consequence not only of the theses 1, 2, 3, 45, but also of these theses and the thesis I, as: CC3CC223CCC3233 = CC3C13CC033 = CC33C13 = C13 = 3. Therefore, the thesis XXXII is not a consequence

of the, so called, positive logic, i.e., of the theses I, 1, 3.

But the thesis XXXII satisfies our matrix. It shows that the degree of extension of this system is higher, than third.

4.8 Recently, A. Church has published his "weak implicational propositional calculus" determined by four axioms which are our theses 3,1,11,45. Evidently, this system is a part of my system presented in this paper. Therefore, my Theorem I holds also for this calculus of Prof. A. Church. From §1 and §4.2 we can establish that a set of theses (3,1,11,45) is inferentially equivalent with (1,2,3,45) or (12,2,3,45) and from §4.7 that the degree of extension of this weak implicational calculus is higher, than third.[13]

NOTES

1) In this paper I use the well known symbolism of Prof. J. Łukasiewicz. The explanation of this symbolism can be found, e.g., in Łukasiewicz$_1$, pp. 77-83.

2) Cf. Sobociński$_1$, §4.

3) Cf. Lewis-Langford$_1$, pp. 85-89. Only four theses of this work (7.52, 7.64, 7.65, 7.92), "paradoxical" in the opinion of the Authors, hold in this system; cf. §4.6 of this paper.

4) Evidently, from this moment the letters: a, b, c, d, f_1, f_2, F_1, F_2, etc. cannot be used, as symbols of any functor or concrete variable.

5) The proof of this theorem presented below is the same, as in Sobociński$_1$, §4.

6) This result is due to J. Łukasiewicz and was published in Sobociński$_1$, pp. 184-185. I use the term "degree of extension" in the sense of the Definition 6 given by A. Tarski in his paper Tarski$_1$, p. 6.

7) It seems to me that this result is due to J. Łukasiewicz, but was never published. The matrices A and B are mine.

8) Cf. Łukasiewicz-Tarski$_1$, §3 and also Łukasiewicz$_2$.

9) Cf. Wajsberg$_1$.

10) Cf. Słupecki$_1$.

11) Cf. Sobociński$_2$. In this paper I have given a proof of the axiomatization of n-value complete systems of calculus of propositions for any integer n. These systems defined in some special and uniform way are in some sense the strongest possible. The axiomatic given above is a simple application of the general axiomatic for the case of a three-value system, i.e., for $n = 3$. Instead of the thesis VIII I used in Sobociński$_2$ some other thesis, but at that time the result of J. Łukasiewicz that VIII is the shortest axiom of the implicational calculus was not known, cf. Łukasiewicz$_3$.

The theorem given above is a direct conclusion from my results included in Sobociński$_2$, but was never before published explicitly. The explanation of this theorem requires a more fundamental discussion. I intend to publish a special paper on this question.

12) Cf. Lewis-Langford$_1$, pp. 85-89 and also Feys$_1$, p. 481.

13) Cf. $Church_1$ and $Church_2$. The last matrix from $Church_2$, p. 30 was used by Mr. A. Church independently from my past results (a letter to the author, Princeton, N. J., 4/30/1952).

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